(3) "Three homographic maps," by Professor FRANK MORLEY, Johns Hopkins University.

(4) "Freshman mathematics for non-technical students", by Professor R. F. BORDEN, George Washington University.

The afternoon session was devoted to relativity. At the invitation of the section Dr. PAUL R. HEYL of the Bureau of Standards spoke on physical aspects of relativity and Professor F. D. MURNAGHAN of Johns Hopkins University on mathematical aspects. Dr. G. Y. RAINICH of Johns Hopkins University led the discussion which followed.

Abstracts of the papers:

1. Mr. Willis gave formulae for the rapid interpolation in the case of certain species of functions where the values of their several members can be represented by the equations

$$F_i(x_n) = M \cdot F_i(x_0) + N \cdot F_i(x_1) .$$

The F's may be any functions whose second derivatives are respectively proportional to them at a given value of the argument. In the computation, $M = m + \Delta m$ and $N = n + \Delta n$. In particular, formulae were given for the cases in which the F's were the rectangular coordinates of a body in orbital motion, $\cos x$ and $\sin x$, or $\cosh x$ and $\sinh x$.

2. Mr. Woolard gave a discussion of the more successful attempts thus far made to apply both graphical and numerical statistical methods to long range forecasting and to the elucidation of the relations between weather at different times and places over the globe. It was concluded that statistical methods have so far been relatively unproductive in meteorology; and that the outlook for the future, particularly as regards the weather problems of the temperate zone, is not hopeful.

3. Professor Morley stated that the homography, $axy + bx + cy + d = 0$ is considered as a correspondence between the points of two maps, M_1, M_2, lying face up in a plane. There are two fixed points and a measure. For three maps, M_1, M_2, M_3, we have 3 homographies whose resultant is identity, 3 pairs of fixed points and 3 measures. On investigating the question as to whether the fixed points determine the 3 homographies, it is found that in general they do.

4. Professor Borden raised the question of what is a desirable course for freshmen who enter with only one year of algebra and take only one year of mathematics in college. He considered it more desirable to give a fairly wide range of ideas and methods with simple applications to clarify them, than to emphasize manipulation of intricate exercises in a narrower field of topics. A suggested course included the usual algebra and trigonometry with em-

phasis on graphical representation, and some elements of the calculus worked in where they could be used to advantage.

5. Dr. Heyl's subject was "The present experimental status of the theory of relativity." The theory of relativity was not founded solely upon the negative result of the Michelson-Morley experiment but upon the joint testimony of a number of different experiments of the same nature. In consequence, if it is now found that the Michelson-Morley experiment gives a positive result we must suspend judgment until these experiments are repeated under the same conditions which apparently altered the result of the Michelson-Morley experiment, namely at a great altitude.

There are six different lines of experiment to be considered. The three confirmatory tests originally suggested by Einstein have now all given a result favorable to his theory. The last of these came into line only recently, that is the shift of the rays of the spectrum, which has lately been confirmed very satisfactorily by means of the companion star of Sirius. The Michelson-Morley experiment appears at present to give a result unfavorable to Einstein while the other two experiments are favorable to Einstein but must be repeated at different altitudes before judgment is reached.

6. In discussing the mathematical aspects of the theory of relativity, Professor Murnaghan called attention to the serious difficulties which confront the student who is introduced to the subject in its historical order of development. In the "special" theory it is perfectly easy to develop the time idea since the assumption as to the constancy of the velocity of light is made. In the general theory no such *a priori* discussion of time can be made and we are forced to an *a posteriori* position. In remarking on Silberstein's discussion of Professor Miller's recent experiments, the speaker pointed out that there is at present an important problem of somewhat the same type outstanding in hydrodynamics; namely, the viscous layer on a body in motion in a fluid. It is entirely possible that a discussion of one of these problems may help to clarify the other although they arise in fields apparently so remote.

J. A. BULLARD, *Secretary-Treasurer*

A NEW METHOD FOR DETERMINING A SERIES SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT OR VARIABLE COEFFICIENTS[1]

By W. O. PENNELL, Southwestern Bell Telephone Co., St. Louis, Mo.

1. Introduction. This paper describes a method for determining a power series solution of linear differential equations with constant or variable coefficients by a method of operational division. Operational methods for the solution of linear equations with constant coefficients are well known having been developed by Oliver Heaviside and many others.[2] The method outlined in this paper is a general method which is applicable not only to equations with constant coefficients but also to equations with variable coefficients, and includes Heaviside's method as a special case. So far as the present investigation has developed it appears that this method of solution is, in general, applicable to all types of linear differential equations. Cases may arise where the method involves the solution of an algebraic equation of degree higher than the second for which only a numerical approximation can be obtained.

The method, in general, consists of replacing the differential notation d^n/dx^n by p^n in the differential equation and then obtaining the value for the dependent variable by an algebraic solution which is called the operational solution. The series solution is then obtained by a method of operational division which is the key to the process.

2. Notation. In this paper the following notations will be used: $p(\ )$ denotes $d/dx(\ )$, $p^n(\ )$ denotes $d^n/dx^n(\ )$, $1/p(\ )$ denotes $\int(\ )dx$. $1/p^2(\ )$ denotes $\int\int(\ )\,dxdx$ etc. For example, px^n is interpreted as nx^{n-1} $1/p$ is x, x^n/p is $x^{n+1}/(n+1)$, $(1/p)e^x$ is e^x, etc. Attention is called to the fact that in this definition of $1/p$ the constant of integration is omitted.

The symbol $\backsimeq$ will be used to denote operational equivalence. For example, $(1/p)\backsimeq x$, $(1/p^n)\backsimeq(x^n/n!)$. Equations of operational equivalence are not subject to the laws of ordinary algebraic equations.

3. Linear equations with variable coefficients. *Simple operational division.* The operational solution of the differential equation is first obtained

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[1] Read at the Kansas City meeting of the Association, December 31, 1925. This paper has been confined to ordinary linear differential equations. The methods are also applicable to partial linear differential equations. The operational division described, or an adaptation from it, is applicable to many non-linear differential equations. The writer wishes to acknowledge his indebtedness to Mr. H. R. Fritz who has read the paper and made helpful suggestions; also to Professor W. H. Roever, of Washington University, for many helpful criticisms and comments.

[2] The underlying principles of Heaviside's work as well as those involved here are symbolic in character. A fairly complete bibliography by Mr. Eugene Stephens, of the literature on symbolic methods, is found in *Washington University Studies, volume XII, Scientific Series no. 2*, pp. 137–52, 1925.

and then the algebraic solution is found by a process of operational division. This can perhaps best be made clear by a simple example.

Take the equation

$$\frac{dy}{dx}+x^{1/2}y=1\,,\quad \text{or}\quad y\eqsim\frac{1}{p+x^{1/2}}\,. \tag{1}$$

(1) is the operational solution. To obtain the actual solution, divide the right hand member of (1) by long division as follows:

$$\begin{array}{l|l}
p+x^{1/2} & 1 \qquad\qquad \left|\ \dfrac{1}{p}-\dfrac{x^{3/2}}{p}+\dfrac{2}{5}\dfrac{x^3}{p}-\dfrac{2x^{9/2}}{4.5p}+\cdots\right. \\
\quad 1+x^{3/2} & \\
\quad -x^{3/2} & \\
\quad -x^{3/2}-\dfrac{2}{5}x^3 & \left.\ x-\dfrac{2}{5}x^{5/2}+\dfrac{2.x^4}{4.5}-\dfrac{2^2x^{11/2}}{4.5.11}+\cdots\right| \\
\qquad\quad \dfrac{2}{5}x^3 & \\
\qquad\quad \dfrac{2}{5}x^3+\dfrac{2x^{9/2}}{4.5} & \\
\qquad\quad -\dfrac{2x^{9/2}}{4.5} &
\end{array}$$

Two lines are given in the quotient the upper being the operational terms obtained by division and the lower the algebraic equivalents which are used when multiplying the second term in the divisor.

$$y=x-\tfrac{2}{5}x^{5/2}+\frac{2x^4}{4.5}-\frac{2^2x^{11/2}}{4.5.11}+\cdots$$

is a solution, *i. e.*, a particular integral. The complete primitive is obtained by adding to this the solution of

$$\frac{dy}{dx}+x^{1/2}y=0\quad\text{which is}\quad y=ce^{(-2/3)x^{3/2}}.$$

Hence the complete primitive is

$$y=ce^{(-2/3)x^{3/2}}+x-\tfrac{2}{5}x^{5/2}+\frac{2x^4}{4.5}-\frac{2^2x^{11/2}}{4.5.11}+\cdots$$

In the above equation (1) the complementary function can also be obtained in an operational manner by putting $py+x^{1/2}y\eqsim cp$, where c is a constant. This gives

$$y\eqsim\frac{cp}{p+x^{1/2}}\,.$$

The result by operational division is the same as above, $y=ce^{(-2/3)x^{3/2}}$.

In general, the complementary function will be obtained by substituting for zero some quantity which is equivalent to zero and using such quantity

as a dividend, the coefficient of y being the divisor in operational division. For example, cp in the above illustration is equal to zero because $cp = c(d/dx)1 = 0$. The details for obtaining these quantities will be given later.

In the operational division the terms of the divisor were arranged in descending powers of p. If we arrange them in ascending powers, the division is as follows:

$$\underline{x^{1/2}+p}\;\Big|\;1 \qquad\qquad \Big|\underline{x^{-1/2}+\tfrac{1}{2}x^{-2}+\tfrac{1}{2}.2x^{-7/2}+\tfrac{1}{2}.2.\tfrac{7}{2}x^{-5}+\cdots}$$

$$\underline{1-\tfrac{1}{2}x^{-3/2}}$$

$$\underline{\tfrac{1}{2}x^{-3/2}}$$

$$\underline{\tfrac{1}{2}x^{-3/2}-\tfrac{1}{2}.2x^{-3}}$$

$$\underline{\tfrac{1}{2}.2x^{-3}}$$

$$\underline{\tfrac{1}{2}.2x^{-3}-\tfrac{1}{2}.2.\tfrac{7}{2}x^{-9/2}}$$

$$\tfrac{1}{2}.2.\tfrac{7}{2}x^{-9/2}$$

$$y = x^{-1/2}+\tfrac{1}{2}x^{-2}+\tfrac{1}{2}.2x^{-7/2}+\tfrac{1}{2}.2.\tfrac{7}{2}x^{-5}+\cdots$$

The above series is divergent.

Usually this operational division will give two formal solutions, a convergent and a divergent series depending upon the arrangement of terms in the divisor and dividend, whether in descending or ascending powers of p.

2. *Equations having two particular integrals.* The above statement regarding two solutions, one convergent and the other divergent, is not universally true. In some cases two solutions are obtained, each convergent, by dividing with terms arranged in ascending or in descending powers of p. These two solutions arise in case there are two particular integrals for the equation. An example will illustrate operational division under such conditions.

Given $xp^2y+y=x^2$, or $y \Rightarrow \dfrac{x^2}{xp^2+1}$.

$$\underline{xp^2+1}\;\Big|\;x^2 \qquad \Big|\;\left(\frac{x^2}{x}\right)\frac{1}{p^2}-\left(\frac{x^3}{x(3)!}\right)\frac{1}{p^2}+\left(\frac{x^4}{x4.3.(3)!}\right)\frac{1}{p^2}-\left(\frac{x^5}{x.5.4.4.3.(3)!}\right)\frac{1}{p^2}+\cdots$$

$$\frac{x^3}{3!}-\frac{x^4}{4.3(3)!}+\frac{x^5}{5.4.4.3.(3)!}-\frac{x^6}{6.5.5.4.4.3.(3)!}+\cdots$$

$$\underline{x^2+\frac{x^3}{(3)!}}$$

$$\underline{-\frac{x^3}{(3)!}}$$

$$\underline{\frac{-x^3}{(3)!}-\frac{x^4}{4.3(3)!}}$$

$$\underline{\frac{x^4}{4.3(3)!}}$$

$$\underline{\frac{x^4}{4.3(3)!}+\frac{x^5}{5.4.4.3.(3)!}}$$

$$-\frac{x^5}{5.4.4.3.(3)!}$$

If we divide the other way:

$$\underline{1+xp^2}\;\Big|\;x^2 \qquad \Big|\underline{x^2-2x}$$

$$\underline{x^2+2x}$$

$$\underline{-2x}$$

$$-2x-0$$

Hence $y=x^2-2x$ is a solution and we have obtained two particular integrals.
The complementary functions are obtained by solving

$$p^2y+x^{-1}y=Ap, \tag{1}$$

and

$$p^2y+x^{-1}y=Bp^2. \tag{2}$$

From (1),

$$y=\frac{Ap}{p^2+x^{-1}}, \tag{3}$$

$$y=A\left[x-\frac{x^2}{1.2}+\frac{x^3}{1.2^2.3}-\frac{x^4}{1.2^2.3^2.4}+\cdots\right].$$

From (2),

$$y=\frac{Bp^2}{p^2+x^{-1}}, \tag{4}$$

$$y=-B\log x\left[x-\frac{x^2}{1.2}+\frac{x^3}{1.2^2.3}-\frac{x^4}{1.2^2.3^2.4}+\cdots\right] \tag{5}$$
$$+B\left[1+x-x^2(\tfrac{1}{4}+1)+x^3\left(\frac{1}{2^2.3}+\frac{1}{2^2.3^2}+\frac{1}{2.3}\right)-\cdots\right].$$

Hence a complete solution is:

$$y=-x+\frac{x^2}{2}+\frac{x^3}{1.2^2.3}-\frac{x^4}{1.2^2.3^2.4}+\frac{x^5}{1.2^2.3^2.4^2.5}-\frac{x^6}{1.2^2.3^2.4^2.5^2.6}+\cdots$$
$$+\left[A-B\log x\right]\left[x-\frac{x^2}{1.2}+\frac{x^3}{1.2^2.3}-\frac{x^4}{1.2^2.3^2.4}+\cdots\right]$$
$$+B\left[1+x-x^2(\tfrac{1}{4}+1)+x^3\left(\frac{1}{2^2.3}+\frac{1}{2^2.3^2}+\frac{1}{2.3}\right)-\cdots\right].$$

3a. *Solutions involving a generating function.* In some cases, by division a geometric series is obtained and the answer is the generating function of such series.

An example is the equation $-np^2y+xp^3y=x$. By dividing,

$$y=\frac{x}{-np^2+xp^3}=\frac{-x^3}{6n}-\frac{x^3}{6n^2}-\frac{x^3}{6n^3}-\cdots=\frac{-\frac{x^3}{6n}}{1-\frac{1}{n}}=\frac{x^3}{6(1-n)}.$$

If we reverse the terms in the divisor,

$$y=\frac{x}{xp^3-np^2}=\frac{x^3}{6}+\frac{nx^3}{6}+\frac{n^2x^3}{6}+\cdots=\frac{\frac{x^3}{6}}{1-n}=\frac{x^3}{6(1-n)}.$$

The generating function $\frac{x^3}{6(1-n)}$ is the same in each case and satisfies the equation.

3b. *Case where the generating function is infinite.* Sometimes the remainder from an operational division is the same as the dividend as shown by the following example:

$$xp^3y-p^2y=12x,\quad y=\frac{12x}{xp^3-p^2}, \tag{1}$$

$$\begin{array}{r|l|l} xp^3-p^2 & 12x & \dfrac{12x}{xp^3} \\ \hline & 12x-12x & \\ \hline & 12x & 2x^3 \end{array}$$

The remainder $12x$ is the same as the dividend and if the division were continued the quotient would be $2x^3+2x^3+2x^3+\cdots$ or infinity.

In all such cases the answer is $k\psi(x)\log x$ where $\psi(x)$ is the first term of the quotient as obtained by straight operational division and k is a constant the value of which is found by substituting $k\psi(x)\log x$ in the differential equation and solving for k.

For example, as applied to the above, the answer is $2kx^3\log x$. Substituting this in the differential equation (1) we find $k=1$ so that $y=2x^3\log x$ is the particular integral.

4. *Application to equations with more than two terms containing* y. The method is applicable to equations with more than two terms containing y.

For example take the equation $p^2y+xpy+y=x$.
The operational solution is

$$y=\frac{x}{p^2+xp+1}.$$

In the process of division all terms in the divisor and dividend should be arranged in descending powers of p. In the above case the result of the division is

$$y=\frac{x^3}{(3)!}-\frac{4x^5}{(5)!}+\frac{6.4x^7}{(7)!}-\frac{8.6.4x^9}{(9)!}+\cdots$$

5. *Compound operational division.* In some examples the simple operational division such as has been described fails[1] and a more involved division is necessary which I have called compound division.

An example is $xp^2y+py+xy=x^2$.
Since x is $1/p$, xp^2 is of the degree 1 in p; that is, the first two terms are of the same degree. It will simplify the division to use p as the first term in the divisor. Where the degree of p is the same in two terms generally the term without x should be used as the first term in the divisor.

$$\begin{array}{l|l}
p+xp^2+x & x^2 \\
 & x^2+2x^2+\frac{x^4}{9} \\
 & \underline{-2x^2-\frac{x^4}{9}} \\
 & \quad -\frac{x^4}{9}-\frac{4x^4}{9}-\frac{x^6}{5^2.3^2} \\
 & \quad \underline{\frac{4x^4}{9}+\frac{x^6}{5^2.3^2}} \\
 & \quad\quad +\frac{x^6}{5^2.3^2}+\frac{6x^6}{5^2.3^2}+\frac{x^8}{3^2.5^2.7^2} \\
 & \quad\quad \underline{-\frac{6x^6}{5^2.3^2}-\cdots}
\end{array}$$

$$\begin{array}{l}
\frac{x^2}{p}-\frac{x^4}{9p}+\frac{x^6}{5^2.3^2p}-\cdots \\
\frac{x^3}{3}-\frac{x^5}{5.3^2}+\frac{x^7}{5^2.3^2.7}-\cdots \\
\frac{x^3}{9}-\frac{x^5}{5^2.3^2}+\frac{x^7}{3^2.5^2.7^2}-\cdots
\end{array}$$

[1] When the ratio of the first terms of two successive dividends is a constant, the simple operational division leads to a series of infinite series. The compound division substitutes for each of these infinite series its generating function.

The particular integral is

$$y = \frac{x^3}{3^2} - \frac{x^5}{3^2 \cdot 5^2} + \frac{x^7}{3^2 \cdot 5^2 \cdot 7^2} - \cdots .$$

If the division is performed as usual the first term of the quotient is $x^3/3$ and the first remainder is $-2x^3 - x^4/3$ (not $-2x^3 - x^4/9$ as shown). The division can then be expressed as follows:

$$\frac{x^2}{p + xp^2 + x} \doteqdot \frac{x^3}{3} - \frac{2x^2}{p + xp^2 + x} - \frac{\frac{x^4}{3}}{p + xp^2 + x},$$

$$\frac{x^2}{p + xp^2 + x} \doteqdot \frac{x^3}{9} - \frac{\frac{x^4}{9}}{p + xp^2 + x}.$$

The first term of the quotient is then $x^3/9$ and not $x^3/3$. $x^3/9$ is then written as the first term in the third line of the quotient. The first remainder is $-x^4/9$ and not $-2x^2 - x^4/3$. The terms $2x^3$ and $-2x^3$ and the similar terms shown in bold face type in the example are, in practice, marked with a circle and disregarded and the division continues.

The process is performed mentally, the multiplication to arrive at the term in the third line of the quotient being obtained by subtracting from one the ratio in question and taking the reciprocal of the result. For example the ratio of $-2x^2$ to x^2 is -2, and $1-(-2)=3$. The multiplier to apply to the first term of the second line of the quotient is $1/3$. Similarly the ratio of $4x^4/9$ to $-x^4/9$ is -4. $1-(-4)=5$ and $1/5$ is the multiplier to apply to the second term of the second line of the quotient, etc.

One complementar · function is given by

$$y \doteqdot \frac{c_1 p}{p + xp^2 + x}.$$

$$\begin{array}{l|l}
p + xp^2 + x \;|\; p & \dfrac{p}{p} - \dfrac{x}{p} + \dfrac{x^3}{2^2 p} - \dfrac{x^5}{2^2 \cdot 4^2 \, p} + \cdots \\
\underline{p + xp^2 + x} & \\
\quad - x & 1 - \dfrac{x^2}{2} + \dfrac{x^4}{2^2 \cdot 4} - \dfrac{x^6}{2^2 \cdot 4^2 \cdot 6} + \cdots \\
\quad - x \mathbf{-x} - \dfrac{x^3}{2^2} & \\
\hline
\quad \mathbf{x} + \dfrac{x^3}{2^2} & - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 \cdot 4^2} - \dfrac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \\
\quad \dfrac{x^3}{2^2} + \dfrac{3}{4}\mathbf{x^3} + \dfrac{x^5}{2^2 \cdot 4^2} & \\
\hline
\quad - \dfrac{3^3}{4}\mathbf{x^3} - \dfrac{x^5}{2^2 \cdot 4^2} & \\
\quad - \dfrac{x^5}{2^2 \cdot 4^2} - \dfrac{\mathbf{5x^5}}{2^2 \cdot 4^2} & \\
\hline
\quad + \dfrac{\mathbf{5x^5}}{2^2 \cdot 4^2} + \cdots &
\end{array}$$

Hence

$$y = \frac{x^3}{3^2} - \frac{x^5}{3^2 \cdot 5^2} + \frac{x^7}{3^2 \cdot 5^2 \cdot 7^2} - \cdots + C_1 \left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right].$$

This is not the complete primitive since it contains only one constant. The remainder of the complementary function is obtained by substituting e^t for x in the equation which then reduces to

$$p^2y+e^{2t}y=0$$

The solution is obtained by operational division of $y \circ\!\!-\!\!\!

Example

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}-2x^{-2}y=0.$$

Substituting x^c, for y we get

$$c(c-1)x^{c-2}+acx^{c-1}-2x^{c-2}=0.$$

Equating the coefficient of x^{c-2} to zero we get $c^2-c-2=0$, $c=2$ or -1.

Hence the exponent of x in the first term of one complementary function must be 2, and in the other complementary function -1. To obtain this put

$$p^2y+apy-2x^{-2}y=A-A\,, \tag{1}$$

and

$$p^2y+apy-2x^{-2}y=2Bx^{-2}-2Bx^{-2}. \tag{2}$$

The solutions are obtained by operational division:

$$y=\frac{A-A}{p^2+ap-2x^{-2}}=A\left[\frac{x^2}{2}-\frac{ax^3}{4}+\frac{3a^2x^4}{5.4.2}-\frac{a^3x^5}{5.4.3}+\cdots\right],$$

and

$$=\frac{2Bx^{-2}-2Bx^{-2}}{p^2+ap-2x^{-2}}=B\left[-x^{-1}-\frac{a}{2}\right].$$

Example

$$(2x+x^3)\frac{d^2y}{dx^2}-\frac{dy}{dx}-6xy=0,$$

The index equation is $2c^2-3c=0$, $c=0$ or $3/2$. Hence we put

$$-py+2xp^2y+x^3p^2y-6xy=Ap, \tag{1}$$

and

$$-py+2xp^2y+x^3p^2y-6xy=(x^{1/2}-x^{1/2})B. \tag{2}$$

Note that p was chosen in equation (1) because the first term in the division

$$y=\frac{Ap}{-p+2xp^2+x^3p^2-6x}$$

is a constant, that is, it contains x with exponent 0. And $x^{1/2}-x^{1/2}$ was chosen in equation (2) because in the division

$$y=\frac{(x^{1/2}-x^{1/2})B}{-p+2xp^2+x^3p^2-6x}$$

the first term is $(Bx^{1/2}/(-p))=-2Bx^{3/2}/3)$ which contains x with the exponent $3/2$.

Another example is

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0.$$

The index equation is $c(c-1)+c=0$; $c=0$, 0.

The first term in one of the series is a constant, and in the other $B\log x$, where B is a constant.

For the series beginning with a constant put

$$y=\frac{Ap}{p+xp^2+1}, \tag{1}$$

$$y=A\left[1-\frac{x^1}{1^2}+\frac{x^2}{1^2.2^2}-\frac{x^3}{1^2.2^2.3^2}+\cdots\right]. \tag{2}$$

For the other series put

$$y=\frac{B(x^{-1}-x^{-1})}{p+xp^2+1}, \tag{3}$$

$$\begin{aligned} y=B\log x&\left[1-x+\frac{x^2}{2^2}-\frac{x^3}{2^2.3^2}+\frac{x^4}{2^2.3^2.4^2}-\cdots\right]\\ &+B\left[2x-\frac{6x^2}{2^2}+\frac{22x^3}{2^2.3^2}-\frac{100}{2^2.3^2.4^2}x^4+\cdots\right]. \end{aligned} \tag{4}$$

The division of equation (3) is interesting and illustrates a condition met with in certain equations. When the operational division of $x^{-1}-x^{-1}$ by $p+xp^2+1$ is performed the quotient is

$$B \log x \left[1 - x + \frac{x^2}{2^2} - \frac{x^3}{2^2.3^2} + \frac{x^4}{2^2.3^2.4^2} - \cdots\right] + B\left[x - \frac{x^2}{2^2} + \frac{x^3}{2^2.3^2} - \frac{x^4}{2^2.3^2.4^2} + \cdots\right] \tag{5}$$

with a remainder which forms the following infinite series

$$B\left[1 - \frac{3x}{2} + \frac{5x^2}{2^2.3} - \frac{7x^4}{2^2.3^2.4} + \cdots\right]. \tag{6}$$

Next the remainder (6) is divided by $p+xp^2+1$ and the quotient is

$$B\left[x - \frac{5x^2}{2^2} + \frac{20x^3}{2^2.3^2} - \frac{94x^4}{2^2.3^2.4^2} + \cdots\right] \tag{7}$$

By adding (7) and (5) the result (4) is obtained. The complete primitive is (2)+(4) or

$$y = (A+B\log x)\left[1 - \frac{x^2}{1^2} + \frac{x^2}{1^2.2^2} - \frac{x^3}{1^2.2^2.3^2} + \cdots\right] + B\left[2x - \frac{6x^2}{2^2} + \frac{22x^3}{2^2.3^2} - \frac{100x^4}{2^2.3^2.4^2} + \cdots\right], \tag{8}$$

4. Equations with exponential or other functions. The solution of certain equations containing exponential (including trigonometric) functions or other functions can sometimes be obtained very readily by operational division.

For example $p^n y+e^x y=e^{2x}$, $$y = \frac{e^{2x}}{p^n+e^x}. \tag{1}$$

$$\begin{array}{l|l} p^n+e^x \;\Big)\; e^{2x} & \dfrac{e^{2x}}{p^n} - \dfrac{e^{3x}}{2^n p^n} + \dfrac{e^{4x}}{2^n.3^n\, p^n} - \cdots \\ \quad e^{2x} + \dfrac{e^{3x}}{2^n} & \\ \hline \quad -\dfrac{e^{3x}}{2^n} & \dfrac{e^{2x}}{2^n} - \dfrac{e^{3x}}{2^n.3^n} + \dfrac{e^{4x}}{2^n.3^n.4^n} - \cdots \\ \quad -\dfrac{e^{3x}}{2^n} - \dfrac{e^{4x}}{2^n.3^n} & \\ \hline \quad \dfrac{e^{4x}}{2^n.3^n} & \end{array}$$

So we get $$y = \frac{e^{2x}}{2^n} - \frac{e^{3x}}{2^n.3^n} + \frac{e^{4x}}{2^n.3^n.4^n} - \cdots \tag{2}$$

The complementary functions are obtained by equating the left side of (1) to k_1p, k_2p^2, k_3p^3, $\cdots$, k_np^n respectively and dividing.

5. Application to linear equations with constant coefficients. All the examples given have been of linear equations with variable coefficients. The methods outlined will apply to equations with constant coefficients, in fact, the process of division is in general much simpler when the coefficients are constant. The complementary functions are obtained by equating the left hand side of the equation to k_1p, k_2p^2, etc. and dividing. If the equation

contains $p^n y$, the complementary functions are obtained by using $k_1 p$, $k_2 p^2$, $\cdots$, $k_n p^n$ successively as the dividends.

For example, $p^n y + y = 1$.
The complementary functions are found by evaluating

$$\frac{k_1 p}{p^n+1}, \quad \frac{k_2 p^2}{p^n+1}, \cdots, \frac{k_n p^n}{p^n+1}.$$

6. Theory: The reader has probably noted the great similarity between the operational division and ordinary long division of algebra. In algebra if we divide with the divisor and dividend arranged in terms of descending powers of x we get one answer and if the division is with terms arranged in ascending powers of x we get another answer, generally one series is divergent and the other convergent for certain values of x, provided that the quotient is not finite. The same was true of the operational division.

We saw in the operational division, especially the compound division, that we could substitute the generating function for a series in the quotient and the result would satisfy the equation even though the series for which the generating function was substituted was divergent.

The same is true of ordinary division in algebra.

Take the equation $xy-y=x$ or $y=x/(x-1)$. By division we get

$$= 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$$

or

$$y = -x - x^2 - x^3 - x^4 - \cdots$$

according to the arrangement of terms in the divisor. Each series is divergent for certain values of x, but the generating function of each series is $x/(x-1)$ and it, of course, satisfies the equation $xy-y=x$.

These remarks are to pave the way for the theory which indicates that the operational division merely follows the rules of an algebra whose multiplication table is closely allied to the integration and differentiation of calculus.

Suppose we have an algebra whose multiplication table is expressed by

$$x^a \odot x^b = \frac{x^{a+b}(a)!\,(b)!}{(a+b)!} \tag{1}$$

where the symbol $\odot$ is used to indicate this particular type of multiplication as distinguished from the multiplication of ordinary algebra.

In (1), the symbol $(a)!$ is intended to mean the gamma product of a. If a is an integer it is the same as factorial a.

The operation in (1) is evidently commutative, for

$$x^b \odot x^a = \frac{x^{b+a}(a)!\,(b)!}{(a+b)!} = x^a \odot x^b = \frac{x^{a+b}(a)!\,(b)!}{(a+b)!}.$$

It will be shown later that the operation is also distributive. That is $x^a \odot (x^b + x^c) = x^a \odot x^b + x^a \odot x^c$.

The operation is also associative for

$$(x^a \odot x^b) \odot x^c = \frac{x^{a+b}\,(a)!\,(b)!}{(a+b)!} \odot x^c = \frac{x^{a+b+c}(a)!\,(b)!\,(c)!\,(a+b)!}{(a+b)!\;(a+b+c)!} = \frac{x^{a+b+c}(a)!\,(b)!\,(c)!}{(a+b+c)!},$$

and

$$x^a \odot (x^b \odot x^c) = x^a \odot \frac{x^{b+c}\,(b)!\,(c)!}{(b+c)!} = \frac{x^{a+b+c}\,(b)!\,(c)!\,(a)!\,(b+c)!}{(b+c)!\;(a+b+c)!} = \frac{x^{a+b+c}(a)!\,(b)!\,(c)!}{(a+b+c)!}.$$

The proof can easily be extended to products of more than three terms.

Division is defined, as in ordinary algebra, as the reverse of multiplication. As a consequence of the definition of multiplication it follows that we may write

$$\frac{(x^{a+b})!\,(a)!\,(b)!}{(a+b)!\;x^a \odot} = x^b$$

or, replacing $a+b$ by m and a by n,

$$\frac{x^m}{x^n \odot} = \frac{(m)!\;x^{m-n}}{(n)!\,(m-n)!}. \tag{2}$$

In other words, x^m in this algebra divided by x^n equals

$$\frac{(m)!\;x^{m-n}}{(n)!(m-n)!}.$$

Constants are merely scalar multipliers as in ordinary algebra.

Addition and subtraction are the same as in ordinary algebra.

Now if a is a positive integer

$$\frac{x^a}{(a)!} \odot x^n = \frac{x^{a+n}(a)!\,(n)!}{(a)!\,(a+n)!} = \frac{x^{a+n}}{(n+1)(n+2)\cdots(n+a)}.$$

But

$$\frac{1}{p^a}\,x^n = \frac{x^{a+n}}{(n+1)(n+2)\cdots(n+a)} \tag{3}$$

for all fractional values of n, and for all positive integral values of n, and for all negative integral values of n, provided $|\,a\,| < |\,n\,|$.

Again if a is a positive integer

$$\frac{x^{-a}}{(-a)!} \odot x^n = \frac{x^{-a+n}(-a)!\,(n)!}{(-a)!\;(-a+n)!} = n(n-1)\cdots(n-a+1)x^{n-a}.$$

But

$$p^a x^n = n(n-1)\,(n-2)\cdots(n-a+1)x^{n-a}. \tag{4}$$

If a is fractional, then multiplication by $(x^a/a!)\odot$ defines fractional integration and multiplication by $(x^{-a}/(-a)!)\odot$ defines fractional differentiation.

We see therefore that with the limitation given above multiplying by $(x^a/(a)!)\odot$ is equivalent to operating by $1/p^a$ and multiplying by $(x^{-a}/(-a)!)\odot$ is equivalent to operating by p^a.

The exception as outlined above where this multiplication does not parallel the operation of calculus may be explained more clearly by an example

$$\int x^{-1}dx = \log x$$

but
$$x\odot x^{-1} = (-1)! = \infty = \int_0^x x^{-1}dx.$$

In other words, where the integration of dx/x is involved, the multiplication $x\odot x^{-1}$ gives not the indefinite integral, but the definite integral $\int_0^x (dx/x)$.

The introduction of the constant of integration is also interesting. We have

$$\int x dx = \frac{x^2}{2} + c.$$

The parallel operation is $x\odot x$ but this can be written $x\odot\left[x + \frac{cx^{-1}}{(-1)!}\right]$ since $\frac{cx^{-1}}{(-1)!} = 0$.

$$x \odot x = \frac{x^2}{(2)!} + \frac{c(-1)!}{(-1)!} = \frac{x^2}{(2)!} + c.$$

Since the multiplication of this algebra parallels integration and differentiation and since the two latter processes are distributive it follows that the multiplication of this algebra follows the distributive law.

Now I will give some examples of algebraic division and operational division.

Pure algebraic division

$$\begin{array}{l|l} x-x^2 & \dfrac{x^2}{x} + \dfrac{x^3}{x} + \dfrac{x^4}{x} + \dfrac{x^5}{x} + \cdots \\ \dfrac{x^2 - x^3}{x^3} & x + x^2 + x^3 + x^4 + \cdots \\ \dfrac{x^3 - x^4}{x^4} & \\ \dfrac{x^4 - x^5}{x^5} & \end{array}$$

Therefore $y=x+x^2+x^3+x^4+\cdots$ or its generating function $y=x/(1-x)$ is a solution of $xy-x^2y=x^2$.

In the above division two lines are shown in the quotient, the indicated division and the actual result. This is done in order to bring out the correspondence between the algebraic and operational division.

If the terms in the divisor are arranged in descending powers of x the result is

$$y = \frac{x^2}{-x^2+x} = -1 - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \cdots$$

This series also formally satisfies the equation $xy-x^2y=x^2$.

Pure operational division:

$$\begin{array}{l|l|l} x\odot-x^2\odot & x^2 & \dfrac{x^2}{x\odot}+\dfrac{2x^3}{3x\odot}+\dfrac{2x^4}{6x\odot}+\dfrac{2x^5}{3.5x\odot}+\cdots \\ & x^2-\dfrac{2x^3}{3} & \\ & \dfrac{2x^3}{3} & 2x+2x^2+\dfrac{4x^3}{3}+\dfrac{2x^4}{3}+\cdots \\ & \dfrac{2x^3}{3}-\dfrac{2x^4}{6} & \\ & \dfrac{2x^4}{6} & \\ & \dfrac{2x^4}{6}-\dfrac{2x^5}{3.5} & \end{array}$$

Therefore $y=2x+2x^2+(4x^3/3)+(2x^4/3)+\cdots$ is a solution of $x\odot y-x^2\odot y=x^2$ or

$$\int ydx-2\int\int ydxdx=x^2.$$

Another example:

$$\begin{array}{l|l|l} \dfrac{x^{-1}}{(-1)!}\odot+1 & 1 & \dfrac{1}{\dfrac{x^{-1}}{(-1)!}\odot}-\dfrac{x}{\dfrac{x^{-1}}{(-1)!}\odot}+\dfrac{x^2}{\dfrac{2x^{-1}}{(-1)!}\odot}-\cdots \\ & 1+x & \\ & -x & x-\dfrac{x^2}{(2)!}+\dfrac{x^3}{(3)!}-\cdots \\ & -x-\dfrac{x^2}{2} & \\ & \dfrac{x^2}{2} & \end{array}$$

Therefore $y=x-(x^2/(2)!)+(x^3/3!)-\cdots$ is a solution of $(x^{-1}/(-1)!)\odot y+y=1$ or $(dy/dx)+y=1$.

Mixed algebraic and operational division:

$$\begin{array}{l|l|l} x\cdot-x^2\odot & x^2 & x+\dfrac{x^2}{3}+\dfrac{x^3}{3^2.2}+\dfrac{x^4}{5.4.3^2}+\cdots \\ & x^2-\dfrac{x^3}{3} & \\ & \dfrac{x^3}{3} & \\ & \dfrac{x^3}{3}-\dfrac{x^4}{3^2.2} & \\ & \dfrac{x^4}{3^2.2} & \\ & \dfrac{x^4}{3^2.2}-\dfrac{x^5}{5.4.3^2} & \\ & \dfrac{x^5}{5.4.3^2} & \end{array}$$

Therefore $$= x+\frac{x^2}{3}+\frac{x^3}{3^2.2}+\frac{x^4}{5.4.3^2}+\cdots$$

is a solution of

$$xy-2\int\int ydxdx=x^2.$$

The operational algebra outlined differs from the calculus under certain conditions. If, however, we had defined the multiplication table as

$$x^a \odot x^b = \frac{x^{a+b}(a)!\,(b)!}{(a+b)!} \tag{1}$$

with the exception that, when a and b are positive integers and b is not greater than a,

$$x^a \odot x^{-b} = a!\ \underbrace{\int\int \cdots \int\int}_{a \text{ times}} x^{-b}dx \cdots dx,$$

$x^{-b} \odot x^a$ being defined as in (1), then the algebra and the calculus will be exactly parallel. With this definition of multiplication the algebra is of course not always commutative as, for example, $x \odot x^{-1} = \log x$ but $x^{-1} \odot x = (-1)!$.

When this is done it is easier to think of the rules of integration and differentiation when performing the operational division, than to remember the rule of multiplication in (1), and the rule of division outlined elsewhere.

7. Conclusion. Let me conclude with a general proof of the

THEOREM: *If the result of operational division is a polynomial or a convergent infinite series, then it is a solution of the differential equation.*

PROOF: Let

$$[p^n + \psi(x)p^{n-1} + \phi(x)p^{n-2} + \cdots + \theta(x)]y \equiv (p^n + p_s)y = \Delta \tag{1}$$

be any linear differential equation, where Δ represents any function of x, and where, for convenience,

$$p_s = \psi(x)p^{n-1} + \phi(x)p^{n-2} + \cdots \theta(x)\,.$$

Then

$$y = \frac{\Delta}{p^n + p_s} \tag{2}$$

and can be found by operational division as follows:

$$\begin{array}{l|l}
\underline{p^n + p_s} \;\Big|\; \Delta & \left| \begin{array}{l} \dfrac{\Delta}{p^n} - \dfrac{1}{p^n}p_s\dfrac{\Delta}{p^n} + \left(\dfrac{1}{p^n}p_s\right)^2\dfrac{\Delta}{p^n} + \cdots \\ \\ \text{----} + (-1)^{r-1}\left(\dfrac{1}{p^n}p_s\right)^{r-1}\dfrac{\Delta}{p^n} + \text{----} \end{array} \right. \\
\dfrac{\Delta + p_s\dfrac{\Delta}{p^n}}{-p_s\dfrac{\Delta}{p^n}} & \\
\dfrac{-p_s\dfrac{\Delta}{p^n} - p_s\dfrac{1}{p^n}p_s\dfrac{\Delta}{p^n}}{+p_s\dfrac{1}{p^n}p_s\dfrac{\Delta}{p^n}} &
\end{array}$$

In the quotient $(-1)^{r-1}((1/p^n)p_s)^{r-1} \cdot \Delta/p^n$ represents the rth term and the expression $((1/p^n)p_s)^{r-1}$ indicates that the operator $(1/p^n)p_s$ is repeated $r-1$ times.

Now we know that the formal process represented by this operational division can be performed. We know that in many cases the result is a convergent series and in cases we know the operational division gives a divergent series. Let us consider the case where the result is a convergent series.

Substitute the first r terms of the quotient in the left side of (1) and denote the result by S_r. Then

$$S_r=\Delta-p_s\frac{\Delta}{p^n}+p_s\left(\frac{1}{p^n}p_s\right)\frac{\Delta}{p^n}-\cdots+(-1)^{r-1}p_s\left(\frac{1}{p^n}p_s\right)^{r-2}\frac{\Delta}{p^n}$$

$$+p_s\frac{\Delta}{p^n}-p_s\left(\frac{1}{p^n}p_s\right)\frac{\Delta}{p^n}+\cdots+(-1)^r\,p_s\left(\frac{1}{p^n}p_s\right)^{r-2}\frac{\Delta}{p^n}$$

$$+(-1)^{r-1}\,p_s\left(\frac{1}{p^n}p_s\right)^{r-1}\frac{\Delta}{p^n}$$

. . . . . .

$$S_r=\Delta+(-1)^{r-1}\,p_s\left(\frac{1}{p^n}p_s\right)^{r-1}\frac{\Delta}{p^n}.$$

Since by hypothesis we are considering only the case when the quotient is a convergent series, it follows that as r is indefinitely increased that $((1/p^n)p_s)^{r-1}\cdot\Delta/p^n$ approaches zero as its limit and $(-1)^{r-1}p_s((1/p^n)p_s)^{r-1}\cdot\Delta/p^n$ will also approach zero as a limit provided p_s is not an operator which will change an infinitesimal into a finite quantity.

We have then, in case the series is convergent and we substitute an infinite number of its terms in the equation, $\lim_{r\to\infty} S_r=\Delta$, and the equation is satisfied.

Since p_s is $\psi(x)p^{n-1}+\phi(x)p^{n-2}+\cdots+\theta(x)$, the operator p_s in general will be a finite operator, unless one or more of the functions $\psi(x)$, $\phi(x)$, etc., are infinite for certain values of x. For such values of x the remainder may not approach zero as the number of terms is indefinitely increased and the operational division then will not satisfy the equation.

The proof for the complementary functions is similar except that for Δ are substituted respectively A_1p, A_2p^2, $\cdots$, A_np^n, giving n complementary functions.

A NEW TYPE OF FRESHMAN COURSE IN MATHEMATICS

By N. J. LENNES, University of Montana

In view of the present somewhat unsettled condition of courses in mathematics for college freshmen, a brief account of an experiment now under way in the University of Montana may be of sufficient interest to warrant publicity. The idea which led to the new course to be described in this paper arose

from experience with a definite concrete case, which is described in more detail in the May number, 1914, of the *Educational Review*, under the caption "Mathematics for Culture." In that paper there is given also a synopsis of a possible course to meet the demand presented by this particular case. In the main the course now being given at the University of Montana follows that synopsis.

In brief, the case was that of a young man of fine intelligence and high educational purposes whose ultimate aim was the law, the profession of his father, but who wanted to spend four years in college as an intellectual "sporting proposition." He came to the department of mathematics in one of the great American universities with the request that he be given an opportunity to learn something about the nature of mathematics, what it has done in developing our present civilization both on its material and intellectual (also spiritual) sides, and what it is continuing to do in man's present inroads into the realm of the unknown. He was not afraid of hard work or difficult subjects, and was possessed of abundant intellectual curiosity. The response to this request was an offer of two years of college mathematics ending with the calculus. "Anything less would be far too fragmentary and entirely inadequate for your purpose. More would be better." But this was much more time than this young man could judiciously give to the subject. It would consume at least one-eighth of his college course—possibly one-sixth. There were many other departments with equal claims on his attention and his curiosity was at least as keen about the mysteries of modern physics and chemistry as about mathematics. The atom had been broken up! What are electrons and protons and "quanta," which he had seen mentioned in semi-popular scientific writings? What is the present status of the doctrine of evolution? What about heredity and bacteria and the geological history of the earth and the recent investigations into the extent of the sidereal universe? Certainly he could not devote two years, even a few hours a week, to each of the departments in which he might hope for light on these varied subjects. Besides, it was necessary that a large share of the four year course should be devoted to the subjects nearer his proposed life work. History, economics, politics in the more general sense, were entitled to more time than the sciences. Besides all these he should study at least one foreign language and possibly some of the world's great literary productions.

The upshot of it all was that the university had nothing to offer in mathematics which this young man could afford to take. A course in trigonometry or in college algebra would be entirely too fragmentary and too barren of results for his purpose.

The question then presented itself as to whether this state of affairs is, in the nature of the case, a necessary evil or whether it might be remedied.

Certainly this young man represents a class of the highest type, to whose wants the college should seek to minister if that is at all possible. It may be granted for the sake of the argument that our traditional sequence of courses is the best that can be arranged for those who are to devote a large share of their college work to mathematics, but may it not be possible to organize a course which should meet the requirements of our somewhat precocious freshman? Is it possible that in our interest in our subject and in our more or less conscious zeal to "build up our department" in the sense of attracting students to the more advanced courses, we have neglected entirely legitimate demands on the part of those whose center of interest lies quite remote from our subject?

With this situation in mind and with a feeling that the correct answers to the questions just stated may very likely be in the affirmative, it was decided some years ago by the members of the department of mathematics in the State University of Montana to offer a course entitled "A Survey of College Mathematics" to be open to all freshmen who enter with at least two years of preparatory mathematics—one year of algebra and one year of plane geometry. The course was first given in the autumn quarter of 1924 and has been repeated each quarter since then, including the summer quarter of 1925, so that now it is in its seventh quarter.

The course is based upon a manuscript text which has been mimeographed for student use. This text consists of five chapters whose titles will give a first indication of the content of the course. I. Trigonometry: indirect measurement of distances; II. Functions and their graphs; III. Derivatives and their uses; IV. Integrals and their uses; V. An historical sketch.

The first chapter confines itself to the definition of trigonometric functions and the proof of the sine- and cosine-formulas. (Logarithms are not introduced.) It is shown that by means of these the triangle may be solved completely and thus distances measured indirectly. It is shown how, by means of these formulas and the necessary tables as a mathematical nucleus, a vast body of interesting information has been obtained and tasks have been performed which without them would be very difficult or altogether impossible. The dimensions of the solar and sidereal systems, the heights and exact locations of mountains that have never been scaled, would be forever beyond us without these formulas or their equivalents. The correct mapping of a great cordilleran system such as that of western North America, or of a long and intricate coastal region with thousands of indentations and islands would be impossible if "triangulating" could not be used, and this requires the use of our two innocent-looking formulas.

In the chapter on functions and their graphs, there is a natural continuation of the subject of graphs as begun in elementary algebra. The Cartesian system is described fully; and then follow in succession brief considerations

of the analytic geometry of the straight line, parabola, circle, ellipse, hyperbola, the cubic, the special exponential 2^x, and the "curve of chance" or "normal distribution." Some of the simple connections of these with phenomena of general interest are pointed out. The planets and comets travel in orbits which are conic sections (probably all ellipses). The relative rapidity with which the functions x^2 and x^3 increase as x increases represents a fundamental fact which limits the size of both animate and inanimate structures, except masses of matter that assume spherical forms under the pressure of gravitational attraction. Biological forms provided with unlimited food and not in any way checked in their increase in numbers, will multiply according to a law represented by a curve of the type a^x (2^x is the one actually studied); and money placed at compound interest and subject to no untoward vicissitudes will increase according to the same law. Examples are given to show the statistical distribution of "traits" in plants and animals and of results which are said to depend upon chance. The equation involved here is of course too difficult for study at this stage and this is stated freely. The geometric series is studied and its use in financial problems is pointed out. The student is informed that the "Mathematical Theory of Investments," which is becoming of increasing importance, depends very largely on the geometric series and its sum; that critical statistical studies in economics, psychology, education, and other subjects are rapidly becoming indispensable, and that in all these the curve of "normal distribution" and its equation are fundamental. In short, the subject of which we are making a first study is in use everywhere about us. It is the heart of the equipment which is of constant use in man's present rapid conquest of the vast unknown.

In chapter III, the derivative is defined, beginning with a critical examination of the meaning of speed at a given instant (point) of time, followed by a similar consideration of acceleration, and then of the concept of tangent to a curve. Derivatives of simple algebraic polynomials and of simple cases involving square roots are found and these are used in writing the equations of tangents to curves and in finding maxima and minima. Derivatives of areas and volumes are also found. It is pointed out that there are many other types of functions whose derivatives are not found, and that there are many other uses to which the derivative may be put besides finding tangents and maxima and minima.

In Chapter IV the problem is to find an expression whose derivative is given. The meaning of constants of integration is studied and problems in finding areas and volumes are solved. The ease with which a volume such as that of the sphere may be found is truly impressive, especially to those who have studied solid geometry. The superiority of the methods of Newton and Leibnitz over those of the Greeks is stupendous. Is there anything in history

which shows more clearly the value of an accumulating body of knowledge and how each step in advance becomes henceforth the heritage of the race?

Certain very simple differential equations are solved such as $(d^2s/dt^2) = -g$. It is pointed out that frequently the world which we are seeking to investigate presents her secrets to us in, what is in effect, the form of differential equations and that these are often so complicated that as yet only approximate solutions are possible, the "problem of three bodies" for instance, being such a one. There is therefore original work still to be done by the keenest among those who are occupying themselves with research in mathematics.

It is pointed out that with the information which the class has about the calculus the area of a circle cannot be found, nor can the length of a single curve be found, nor the area of a surface unless it is flat. Indeed all along the course the limitations of what we are learning are pointed out and problems suggested which could be solved if we were to take time to extend our study, using however essentially the methods with which we are becoming familiar.

The experience of the last seven quarters in this course, with college freshmen presumably of normal intelligence and preparation, has proved beyond a doubt that the subject matter is easily within their reach and that it can be covered in one quarter meeting five hours a week, and of course equally easily in a semester meeting three times a week. This experience has proved also that the course is of real interest to such students. It is the unanimous testimony of those who have taught it that the first year students show greater interest in this course than in other freshman courses in mathematics. This is also shown by the fact that it is being elected by a considerable number of grade A students who are not expecting to specialize in mathematics and who presumably select what they think is worth-while. Our purpose to build a course that should meet the needs of the type of student described at the beginning of this paper seems to have been realized to a reasonable degree. Certain it is that the time required is not beyond what such a student can afford to give to the subject. Not a few have expressed the desire that similar survey courses might be offered in a number of other departments so that those whose first interests lie at the other end of the campus might have an opportunity to acquire a wide range of authentic information about the subject matter, method, and general significance of the various parts of the field of science.

While the satisfaction of the needs of the non-mathematical student was the sole early aim in organizing this course, we did not go far with that work before the question arose as to whether the course might not be the very best possible introduction to college mathematics even for those who are to specialize in this subject or to do enough work in it to satisfy the requirements for physics, chemistry, or general engineering, and we came to the conclusion that an

affirmative answer is the right one. It was fully recognized that there is justification for the feeling of those who espouse the cause of "unified mathematics" that the classical separate courses fail to show the interdependence of these courses, an understanding of which is necessary for adequate comprehension of each. But at the same time, it was believed that the newer "unified" courses have a tendency to be superficial and that they fail to give that continued attention to each of the well defined separate elements of the subject, which is also necessary for adequate comprehension. The question arises as to whether this new arrangement may possibly possess the merits of both older plans and at the same time avoid the pitfalls of each.

Employing an extended figure of speech, let us consider the exploration and study of the geography and geology of an unknown region. The work may conceivably be done in several ways. The geographical features may be studied entirely without reference to the geological structure and a complete topographical map prepared, and then as a "separate course" the geological formations investigated. This would correspond to the old type separate courses. Again each part of the region may be studied completely as the work proceeds, the topographical map prepared and the geological structure deciphered for each part the first time the region is traversed. This would typify the unified course. A third possible mode of procedure would be to make a general preliminary reconnaissance giving fairly accurate ideas of the lay of the land and of the main geological conformations and the agencies which have been at work to make the topographical features what they now are. This would be followed by intensive and fairly separate study of the topography, and the geological formation and history. This corresponds to the plan we are now describing. The arguments for this plan appealed so strongly to us that the general survey course has been made a prerequisite for all other courses in mathematics except intermediate algebra.

The students who have had this course come to the more specialized courses in trigonometry, analytic geometry, and calculus with considerable insight into their purpose and method. They know in a general way the character of the whole region and each separate part has a significance which is conducive to a higher type of work. The methods of these subsequent courses are known sufficiently so they can be used to some extent where they are of service in any of these courses. Thus, for instance, derivatives can be used in analytic geometry.

Finally, I should like to register the personal opinion that a course closely similar to this survey course would be the best final course in the high school for those who elect more than two-and-a-half of three years of mathematics and for students in the normal schools. In European schools the ideas of the calculus are brought in much earlier than we have done heretofore. This

course, if given in the high school, would do just that for us, and would give those who do not go to college a much broader outlook on mathematics than they get now even if a half year is devoted to trigonometry, solid geometry, or higher algebra. The analytical part of trigonometry, for instance, has little significance for those who take that subject (and to take it is the present custom), while this survey course is full of meaning and interest throughout.

In this course there is much more attention given to the general human setting of the subject than is customary in our early college courses, at least so far as the character of these is revealed by the texts in use. What was the origin of these methods and formulas—what was their historical setting? What beautiful illustrations we have of how a scientific theory is built! At the time of its formulation as a general theory—the building of an edifice of thought—parts which enter into the structure are taken from many points in the long story whose culmination it is. Some parts may come from India, others from Egypt, Babylonia, Greece, northern Europe. These parts do not constitute the general theory any more than a pile of bricks constitutes a building. But the bricks are necessary. Thus by slow and laborious steps the necessary material is accumulated and in its time placed at the disposal of a genius who perceives "the golden threads that run through all and do all unite" and who thus furnishes the human race with another instrument for investigation. In what ways have the methods and theories which we are studying been of service to the race? What elements in our present technical information and practice are dependent upon them? What do we know about the world in which we live which they have in part been instrumental in discovering? How is the method of thought used in mathematics related to the thinking about other subjects? Such questions are constantly kept to the fore. To the dullest they mean little, but to the more intelligent they are fraught with meaning and may be the starting point of the formation of sane ways of thinking.

Supplementary Note—Certain questions have been raised about this paper which I believe can be considered more appropriately in a supplementary note than by rewriting the paper. "What is the unifying element of the course?" "Is it a lecture course and what is the nature and extent of the students' participation?" "Does not a course of this kind skim the cream off this subject matter, leaving the subsequent more formal courses uninteresting?" "Should not such a course be given to upper class students rather than to freshmen?" "How does this course differ from the 'unified' courses with which we are familiar?"

To reply to these queries in order: The unifying element is the general human significance and interest of the subject matter and the mathematical

method of dealing with it. This has determined the inclusion and exclusion of topics as well as the mode of treatment.

The course is no more of a lecture course than any other course in elementary mathematics. Theorems are developed and exercises and problems solved precisely as in any other such course. If this statement needs any qualification whatever, it may be this, that in making proofs of theorems there is a flavor of mathematical intuition and insight as over against logical rigor. The purely logical element finds full use in applying the theorems in the solution of problems. There is just as much logic necessary in making certain that a fundamental theorem applies in a particular way in dealing with given situations as there is in showing that this fundamental theorem follows from still more fundamental propositions.

If this kind of course renders subsequent courses less interesting by "skimming the cream" off them, the same indictment lies against a great many courses whose propriety we do not question. Limits are considered in geometry, to some extent in algebra (the sum of a series such as $1+\frac{1}{2}+\frac{1}{4}+\cdots$), and in analytic geometry in defining a tangent, while the main treatment of this topic of course comes in the calculus. Some trigonometry is now brought into elementary algebra and geometry; a considerable part of formal analytic geometry is given in a strong course in high school algebra where graphs are stressed; many are urging that derivatives be used in college algebra and also in analytic geometry. My own impression is that interest attaches to that with which we are partly familiar and that this course serves to make the subsequent courses sufficiently familiar to make them more interesting. If this "skimming the cream" objection is valid how *very far* from the right track must be our European colleagues who introduce derivatives comparatively early!

The time at which such a course should be given was considered with some care, having in mind the legitimate demands of other subjects and the departments in which lie the student's main interests. There is a pretty general demand in various departments that certain of their courses be open only to upperclassmen. If this demand is acceded to impartially for all such departments it follows as a necessary consequence that these courses will be available only to a few. As the student advances in his work his interests tend (and I believe properly) to concentrate about his major work and in courses which have a more immediate bearing on the profession for which he is making special preparation. It was this more general point of view, having in mind the college course as a whole, and the many different departments which later come to claim the students main effort which decided us to arrange this course for freshmen. This course was devised with the idea that it should fit in naturally in a scheme in which similar courses are given in a number of departments.

This course differs from the ordinary "unified" course in that it is a "preliminary survey," followed by the standard separate courses in trigonometry, analytic geometry, and calculus. In these later courses there is no unnecessary repetition. There is a brief review and extension of the concepts already studied and then a further study of the main topic. The situation is very much analogous to what we now have in college algebra—a review of the advanced part of high school algebra, the quadratics for example.

The historical material has been placed in a separate chapter to avoid the scrappy notes we find in many texts and to make possible a connected story which shows something of real historical development. This material is used as each topic is considered, and there is a final reading of the whole chapter.

The following questions are from the final examination given at the end of this winter quarter:

In the triangle ABC, $\angle C=62°40'$, $\angle A=49°50'$, $c=1263$ feet. Find the remaining parts of the triangle.

Write the geometric series whose sum represents the accumulated value of an annuity of p dollars per year if the rate of interest is 4.5% and the number of years is 15.

A point moves along the y-axis so that $y=4t^3-6t^2+t-6$. Find a function of t representing its velocity and also a function representing its acceleration. For what values of t will the point be at rest? For what value of t will the acceleration be zero? Find the position of the particle for each of these values of t.

Examine the curve $y=6x^2+3x+8-5x^3$ for maxima and minima. How do you decide which value of x represents a maximum and which represents a minimum?

The curve $y=2x^4-7x^3+11x^2-14x+8$ meets the x-axis in the points $x=1$ and $x=2$. Find the area inclosed between these points by the x-axis and this curve.

Find the values of x for which the tangents to the curves $y=x^2$ and $y=x^3$ are parallel.

Using the method of the calculus, derive the formula giving the volume of a sphere.

ON DIOPHANTINE RELATIONS

By W. H. RITTENHOUSE, Philadelphia, Pa.

1. Theorem. If we have a rational, integral, homogeneous function, $F(x,y)$, and if we put $(\partial F/\partial y)=h$, $(\partial F/\partial x)=k$,

$$F\{(xd-hn),\ (yd+kn)\}=G(x,y,n,d),$$
$$F\{(xd+hn),\ (yd-kn)\}=H(x,y,n,d);$$

then $G(x,y,n,d)$ *will be integrally divisible by* $F(x,y)$, in the same sense that x^m-y^m is in all cases divisible by $(x-y)$. A similar remark applies to $H(x,y,n,d)$.

PROOF:[1] Expand $F(xd-hn,\ yd+kn)$, regarded as a function of six independent variables, in powers of h and k, by Taylor's theorem for functions of two independent variables. The expansion will contain all the various partial derivatives up to those of total order, m, if m is the order of the homo-

[1] This proof was supplied by A. A. Bennett.

geneous function, $F(x,y)$ in x and y. Collecting the terms of total order, r, in h and k, we have as a typical set of terms

$$\sum_i \frac{\partial^r F(xd,yd)}{\partial x^i\,\partial y^{r-i}}\binom{r}{i}(-1)^i(hn)^i(kn)^{r-i}.$$

This set of terms is homogeneous and integral in n and d, individually. Removing the common factor $d^{m-r}n^r$, we have left for examination the sum,

$$S=\sum_i \frac{\partial^r F(x,y)}{\partial x^i\,\partial y^{r-i}}\binom{r}{i}(-1)^i h^i k^{r-i}.$$

We shall show that this sum is always integrally divisible by $F(x,y)$ if h and k are regarded as functions of x and y, by the previous relation, $h=F_y(x,y)$, $k=F_x(x,y)$. By Euler's formula for homogeneous functions, $mF(x,y)=xF_x(x,y)+yF_y(x,y)$. By repetition, we have

$$\binom{m}{r}F(x,y)=\sum_i\binom{r}{i}x^i y^{r-i}\frac{\partial^r F(x,y)}{\partial x^i\,\partial y^{r-i}}.$$

As suggested by Sylvester's dialytic method, we may write out the following sequence of equations:

$$\begin{aligned}
&x^rF_{x^{(r)}}+rx^{r-1}yF_{x^{(r-1)}y}+\binom{r}{2}x^{r-2}y^2F_{x^{(r-2)}y^{(2)}}+\cdots+rxy^{r-1}F_{xy^{(r-1)}}+y^rF_{y^{(r)}}=\binom{m}{r}F,\\
&x^rF_x+x^{r-1}yF_y=mx^{r-1}F,\\
&x^{r-1}yF_x+x^{r-2}y^2F_y=mx^{r-2}yF,\\
&\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&xy^{r-1}F_x+y^rF_y=my^{r-1}F.
\end{aligned}$$

Now each pair of values of x,y, not both zero, which makes F vanish, reduces all the right hand members to zero, and leaves a set of $r+1$ homogeneous linear equations in the $r+1$ quantities, x^r, $x^{r-1}y$, $\cdots$, xy^{r-1}, y^r. The determinant of the coefficients in this set of equations must therefore vanish for such pairs of values. This determinant is exactly the expression, S. Since S vanishes with F, S must contain F as a factor.

2. The theorem above written gives, then, four expressions, G, G/F, H, H/F, which we shall find useful in studying diophantine properties of the function F. The arguments involved are x, y, n, and d, and one may study the variation of these functions as any one or all of the arguments are regarded as varying.

Consider the diophantine relation imposed upon x and y by the equation: $x^3+y^3\equiv f$, (1), where f is a given integer. In accordance with the theorem we have, if we put $F(x,y)\equiv x^3+y^3$,

$$(-y^2n+xd)^3+(x^2n+yd)^3=\{d^3+3xydn^2+(x^3-y^3)n^3\}\cdot f \qquad \text{(I)}$$

$$(y^2n+xd)^3+(-x^2n+yd)^3=\{d^3+3xydn^2-(x^3-y^3)n^3\}\cdot f. \qquad \text{(II)}$$

If $h(x,y)$ and $k(x,y)$ have common numerical factors it is sometimes possible to suppress these if they do not appear as algebraic factors of the function, F, mentioned in the theorem. This remark is made here to prepare the reader should he find such factors omitted. We study then from the diophantine standpoint the relations obtained from the four expressions:

$$X_1=-y^2n+xd,\ Y_1=x^2n+yd,\ X_2=y^2n+xd,\ Y_2=-x^2n+yd. \tag{2}$$

The numerical tabulation of these four expressions, (2), selecting any fixed x and y, and allowing n and d to vary over integral values from zero to as high as one cares to go, will give interesting sets of four numbers, and their relations may be studied from the second members of (I) and (II).

3. Application of X^3+Y^3 to problems in fifth powers. Though we usually think of the cubic relation of X_1, and Y_1 in connection with (I) and (II), there exists the relation in *squares* and *biquadrates*: $X_1^4+Y_1^4-X_2^4-Y_2^4$ is algebraically divisible by $X_2^2+Y_2^2-X_1^2-Y_1^2$ for any X_1, Y_1, X_2, Y_2, satisfying (2). Suppose more generally that we have any four quantities A_1, B_1, A_2, B_2, for which

$$A_1^4+B_1^4-A_2^4-B_2^4=2(A_2^2+B_2^2-A_1^2-B_1^2)q^2\cdot \tag{3}$$

In (3) whether we use X_1, Y_1, X_2, Y_2, or any four integers of which the condition is true, as A, etc., we will always have:

$$(A_1+q)^5+(B_1+q)^5+(A_2-q)^5+(B_2-q)^5=$$
$$(A_1-q)^5+(B_1-q)^5+(A_2+q)^5+(B_2+q)^5. \tag{4}$$

This may be seen from the expansion of (4). In the numbers of the left-hand members of (2) to make q^2 a rational square in (3) we must solve the relation

$$xyd^2-q^2=n^2(x^5-y^5)/(x-y)\cdot \tag{5}$$

Given an x and y this is not always solvable, but solvable cases exist. For example: if $x=2$ and $y=1$, this becomes $2d^2-q^2=31n^2$, which is solvable, and solvable conditions may be obtained at least frequently if we make xy a square, by selecting x and y as squares, 1, 4, 9, etc. We then solve by finding X_1, Y_1, X_2, Y_2, of (2), writing finally the eight numbers of (4).

In one's search for such numbers, A_1, B_1, A_2, B_2, it is merely noted that the four expressions of (2) fulfill the condition. It is doubtful whether any of the eight numbers (4) obtained by using (2) will become zero, or whether two may be equated, in opposite members, but the solution of the reduced problems in which the number of terms in one or both members is reduced may be easily obtained by using expressions slightly different from (2). Take instead:

$$A_1=(-rn+xd),\ B_1=(tn+yd),\ A_2=(rn+xd),\ B_2=(-tn+yd). \tag{6}$$

If these values be substituted in (3), it reduces to

$$\frac{ty^3-rx^3}{rx-ty}d^2-q^2=\frac{r^3x-t^3y}{rx-ty}n^2\cdot \tag{7}$$

The two principal reduced problems will now be treated in detail.

First. To find the sum of three fifth powers equal to four fifth powers, with sums of corresponding first powers also equal, let $A_1+q=0$. Then

$$q=rn-xd\cdot \tag{8}$$

Second. To find the sum of three fifth powers equal to three fifth powers with sums of corresponding first powers also equal, let $A_1+q=B_1-q$. Then

$$q=\tfrac{1}{2}(y-x)d+\tfrac{1}{2}(r+t)n\cdot \tag{9}$$

Select any x and y in integers. Substitute in (7). It is noted that now q^2 in (7) contains a term in d^2 with a square as coefficient, with either (8) or (9). Make the coefficient of d^2 in (7) equal to this square. From (9) we write:

$$\frac{(ty^3-rx^3)d^2}{rx-ty}-\tfrac{1}{4}(y-x)^2d^2-\tfrac{1}{2}(y-x)(r+t)dn-\tfrac{1}{4}(r+t)^2n^2=\frac{(r^3x-t^3y)n^2}{rx-ty} \tag{10}$$

Make the *entire* coefficient of d^2 in (10) zero by writing:

$$t=x^3+\tfrac{1}{4}x(y-x)^2, \tag{11}$$

$$r=y^3+\tfrac{1}{4}y(y-x)^2. \tag{12}$$

As d^2 has vanished, the factor n will divide out of (10), and

$$d=\tfrac{1}{4}(r+t)^2(rx-ty)+(r^3x-t^3y), \tag{13}$$

$$n=-\tfrac{1}{2}(y-x)(r+t)(rx-ty). \tag{14}$$

In (13) and (14) r and t must be written in values of x and y, from (11) and (12), and finally we write q in (9).

Illustration. If $x=3$ and $y=5$, we find, (dropping the common factor 10 from r and t, and -24 from d and n): $r=13$; $t=3$; $d=-333$; $n=16$; $q=-205$. These are the final ratios which may be used, and we obtain:

$$1412+1822+586+1508=1002+1412+996+1918.$$

The fifth powers are also equal, and finally:

$$911^5+293^5+754^5=501^5+498^5+959^5.$$

NOTE: We have followed out in detail and illustration only (9). Equation (8) may be similarly followed out and illustrated, but the results are apparently the same as Haldeman's solution (*1918, 399–402*) of three fifth powers equal to four fifth powers.

MATHEMATICIANS IN THE AMERICAN DICTIONARY OF NATIONAL BIOGRAPHY

A fund of half a million dollars has been established to make possible the publication of an *American Dictionary of National Biography*, comparable with the English work of similar title. The editor is Professor Allen Johnson of Yale University. In order to make the proper selection of names, the collaboration of scientific and historical societies is earnestly desired. A tentative list of American mathematicians (no longer living) to be included in the dictionary is appended. It is understood that mathematicians born abroad are to be classed as American if their principal activities have been with American institutions. Division A includes the names of those supposedly worthy of longer notices; division B includes somewhat doubtful names, or those to have briefer notices.

The Editor of the MONTHLY would appreciate suggestions on this list both as to other names to be included and as to names whose inclusion is

doubtful. Prompt action is desirable. It is hoped, in view of the importance of the proper representation of mathematicians, that many will give the list careful scrutiny and make suggestions. Address all replies to the Editor of the MONTHLY.

Class A

Adams, Daniel (1773–1864)
Adrian, Robert (1775–1843)

Bôcher, Maxime (1867–1918)
Bouton, C. L. (1869–1922)
Bowditch, Nathaniel (1773–1838)
Bowser, Edward Albert (1845–1910)

Chauvenet, William (1820–1870)
Church, Albert E. (1807–1878)
Colburn, Warren (1793–1833)
Courtenay, Edward H. (1807–1853)
Craig, Thomas (1853–1900)

Daboll, Nathan (c.1750–1818)
Davies, Charles (1798–1876)
Day, Jeremiah (1773–1867)
Docharty, G. B. (1804–1889)

Eddy, Henry Turner (1844–)
Evans, Evan William (1827–1874)

Ficklin, Joseph (1833–1887)

Gibbs, Josiah Willard (1839–1903)
Greenleaf, Benjamin (1786–1864)
Greenwood, Isaac (1702–1745)

Hill, George William (1838–1914)
Hill, Thomas (1818–1891)
Halsted, George Bruce (1853–1922)

Jones, Hugh (1669–1760)

Loomis, Elias (1811–1889)

McClintock, Emory (1840–1916)
McCartney, Washington (1812–1856)
Martin, Artemas (1835–1918)

Newcomb, Simon (1835–1909)

Oliver, James Edward (1829–1895)
Olney, Edward (1827–1887)

Patterson, Robert (1743–1824)
Peirce, Benjamin (1809–1880)
Peirce, James Mills (1834–1906)
Peirce, Charles S. S. (1839–)
Pike, Nicholas (1743–1819)

Ray, Joseph (1807–1855)
Rittenhouse, David (1732–1796)
Robinson, Horatio N. (1806–1867)
Runkle, J. D. (1822–1902)

Safford, T. H. (1836–1901)
Sharpless, Isaac (1848–)
Smyth, William (1797–1868)
Strong, Theodore (1790–1869)
Sylvester, James Joseph (1814–1897)

Wentworth, George Albert (1835–1906)
Winthrop, John (1714–1779)
Woodward, Robert Simpson (1849–1924)

Class B

Beman, Wooster Woodruff (1850–1922)
Bledsoe, Albert Taylor (1809–1877)
Buckingham, C. P. (1808–1888)

Davis, E. W. (1857–1918)
Dod, A. B. (1805–1845)
Dodd, James B. (1807–1872)
Doolittle, C. L. (1843–1919)

Eaton, James S. (1816–1865)

Farrar, John (1779–1853)

Grew, Theophilus

Houston, Wm. C. (1748–1788)
Hyde, Edward Wyllys (1843–)

Johnson, William Woolsey (1841–)

Kino, Eusebius

Mansfield, Jared (1759–1830)
McMahon, James (1856–1922)

Newton, Hubert Anson (1830–1896)

Peck, William G. ( )
Pitcher, A. D. (1880–1924)

Quackenbos, Geo. P. (1826–1881)

Shattuck, S. W. (1841–1915)
Smith, Roswell C. (1797–1875)
Spare, John
Stringham, Irving (1847–1909)
Stoddard, John B. (1797–1875)

Thompson, James B. (1803–1883)

Venable, C. S. (1827–1900)

Webber, Samuel (1759–1810)
Wells, Webster (1851–1916)

QUESTIONS AND DISCUSSIONS

EDITED BY TOMLINSON FORT, Hunter College, Park Ave. and 68th St., New York, N.Y.
AND BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate methematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

REPLIES TO QUESTIONS

51 [*1924*, 85; *1925*, 506]. Can any reader supply approximate formulas for the problem of a cable suspended from two points at different levels?

REPLY BY MICHAEL GOLDBERG, Philadelphia, Pa.

This problem is met in the stringing of electric wires and cables in the transmission of electrical energy. The following papers on it have appeared in the *Proceedings of the American Institute of Electrical Engineers*:

"Solutions to Problems in Sags and Spans," Wm. Le Roy Robertson (1911) vol. 30, pages 1111–1129,

"Calculations for Suspended Wires," Percy H. Thomas (1911) vol. 30, pages 1131–1141,

"Mechanical Characteristics of Transmission Lines," H. Pender and N. de K. Thompson, vol. 30, pages 1379–1400.

If the span length is X, the difference of elevation of the supports is h, the maximum tension is T, and the load per foot is w, then x_1, which is the horizontal distance from the upper support to the lowest point in the wire is given by Thomas

$$x_1 = \frac{X}{2} + \frac{h}{K}\left(K - \frac{h}{2}\right),$$

$$x_1 = \frac{\sqrt{d}}{\sqrt{d} \pm \sqrt{d-h}},$$

$K = T/w$

d = distance of lowest point in wire below lower support.

$$x_1 = \frac{X}{2} + \frac{h}{X}K,$$ Molesworth's *Pocketbook*.

If S is the sag or dip for supports at the same elevation, the distance d of the lowest point below the lower support for different elevations is given by

$$d = s\left(1 - \frac{h}{4s}\right).$$ Pender.

A still more accurate formula, which I have used, is given below:

$$x_1 = \sqrt{\left(\frac{3aX}{h}\right)^2 + \frac{3a}{h}(X^2 + 2ah)} - \frac{3aX}{h}$$

where $a = K - h/2$.

DISCUSSIONS

I. On the Vanishing of the Remainder of the Binomial Series

By C. F. Gummer, Queen's University

Let $S_{n,r}(x)$ represent the sum of the finite series $\binom{n}{0}+\binom{n}{1}(x-1)+\binom{n}{2}(x-1)^2+\cdots+\binom{n}{r}(x-1)^r$, where n is a real number. If x lies between 0 and 2, so that $x-1$ is numerically less than 1, the same series continued to infinity converges to the value $\{1+(x-1)\}^n$, or x^n. Whether x lies between these limits or not, let us denote the remainder $x^n-S_{n,r}(x)$ by $R_{n,r}(x)$.

Within the region of convergence, $R_{n,r}(x)$ is the sum of the infinite series $\binom{n}{r+1}(x-1)^{r+1}+\cdots$; and, if x is near enough to 1, the sign of this series is that of its leading term $\binom{n}{r+1}(x-1)^{r+1}$, the first term of the binomial series lacking from $S_{n,r}(x)$. If, for more remote values of x, these signs cease to agree, the change must occur where $R_{n,r}(x)$ becomes either infinite or zero. The infinite cases are found only at $x=0$, for values of n (such as -3) which make x^n infinite there with change of sign; and these need not detain us. The location of the values of x for which $R_{n,r}(x)$ is *zero* forms however a more interesting problem.

We may omit throughout the cases where $R_{n,r}(x)$ vanishes identically in x; these being the cases $n=0, 1, 2, \cdots, r$. In all other cases $R_{n,r}(x)=0$ is an equation having a root of multiplicity $r+1$ at $x=1$. Hence, if $S_{n,r}(x)$ is arranged in powers of x as $c_0+c_1x+c_2x^2+\cdots+c_rx^r$, $R_{n,r}(x)$ must, by Descartes' rule, exhibit at least $r+1$ variations of sign in its coefficients when the powers of x are in numerical order, x^n being inserted in its proper position. But the expression for $R_{n,r}(x)$ contains only $r+2$ terms. Therefore the number of variations must be exactly $r+1$, the coefficients having alternate signs.[1] It follows that $R_{n,r}(x)$ does not vanish for any positive value of x other than 1.[2]

The only real values of n for which the question of *negative* roots arises are of the form p/q, where p is an integer and q a positive odd integer. When x is replaced by $-x$, each variation of sign between consecutive coefficients becomes a permanence, except in some cases where one of the terms is x^n. Hence there cannot be more than one negative root in any case, and the following results are easily verified.

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[1] This fact may be proved directly by finding the values of the c's the result being $c_i=(-1)^{r-i}\binom{n}{i}\binom{n-i-1}{r-i}$.

[2] It is of course easy to see from the finite Taylor development that $R_{n,r}(x)$ has the same sign as $\binom{n}{r+1}(x-1)^{r+1}$ for every *positive* x.

For $n=p/q$, q odd;

(1) If $n>r$, and $p-r$ is even; there is one negative root to $R_{n,r}(x)=0$.
(2) If $n>r$, and : $p-r$ is odd; no negative root.
(3) If $0<n<r$: one negative root.
(4) If $n<0$, p even; one negative root.
(5) If $n<0$, p odd; no negative root.

It is easy to illustrate these statements by drawing the curve $y=R_{n,r}(x)$ in the various cases. To do so it is necessary to note the character of n, the parity of r , which decides whether the curve crosses the x-axis at $x=1$, and, if $n<r$, the sign of $\binom{n}{r}$. It will also be observed that, when p is an even positive integer and q an odd integer greater than p, there is a cusp with vertical tangent where the y-axis is crossed.

When $n=p/q$, p an odd integer and q an even positive integer, the complete algebraic curve has, besides $y=R_{n,r}(x)$, a lower branch $y=T_{n,r}(x)$, where $T_{n,r}(x)=-x^n-S_{n,r}(x)$. This curve has no points to the left of the y-axis. When p is greater than q it has a cusp on the y-axis, with non-vertical tangent.

The question now arises whether the lower branch may have points in common with the x-axis; that is, whether, for certain values of x, $S_{n,r}(x)$ may be equal to the *negative* qth root of x^p. To settle this we may note the signs of $T_{n,r}(x)$ when $x=1$ and as x becomes zero and infinite. From what we already know of $R_{n,r}(x)$, these are readily found to be as in the following table.

		$T_{n,r}(0)$	$T_{n,r}(1)$	$T_{n,r}(\infty)$	No. of roots in $(0,1)$	No. of roots in $(1,\infty)$
(6)	$n>r$, r odd.	+	−	−	1	0
(7)	$n>r$, r even.	−	−	−	*0*	0
(8)	$0<n<r$, r odd, $\binom{n}{r}<0$.	+	−	+	1	1
(9)	$0<n<r$, r odd, $\binom{n}{r}>0$.	−	−	−	*0*	0
(10)	$0<n<r$, r even, $\binom{n}{r}<0$.	−	−	+	*0*	1
(11)	$0<n<r$, r even, $\binom{n}{r}>0$.	+	−	−	1	0
(12)	$n<0$, r odd.	−	−	+	0	1
(13)	$n<0$, r even.	−	−	−	0	*0*

The numbers of roots noted in the two right-hand columns are evidently correct *modulo* 2. Moreover, if $r=0$, which may occur in cases (7) and (13), the figures are exactly true. To show that they are exact in all cases, we may use an inductive argument, using the fact that the derivative of $T_{n,r}(x)$ is $n \cdot T_{n-1,r-1}(x)$. For instance, if there were more than one root in the interval $(0,1)$ under case (6), there would be at least two values in the interval at which the derivative was zero, so that case (7) would be incorrect for a smaller value of r. This argument will be found sufficient for each case, except where

the zeros are given in italics; and these zeros are evidently correct, since here $R_{n,r}(x)$ (and therefore also $T_{n,r}(x)$) is negative throughout.

Some numerical instances follow of the roots of $R_{n,r}(x)=0$ and $T_{n,r}(x)=0$ under the various cases.

(1) $R_{3,1}(-2)=0$; *i. e.* $1+3(-3)=(-2)^3$.

More generally, if n is a positive integer, and $x=-n+1$, then $x^n=S_{n,n-2}$.

(3) $R_{2/3,1}(-1/8)=0$; *i. e.* $1+\frac{2}{3}(-1\frac{1}{8})=(-1/8)^{2/3}$.

(4) $R_{-2,r}(x)=0$ if $x=-1/(r+1)$.

(6) $T_{3/2,1}(1/4)=0$; *i. e.* $1+\frac{3}{2}(\frac{1}{4}-1)=-(1/4)^{3/2}$.

(8) $T_{3/2,3}(7\pm 4\sqrt{3})=0$.

(10) $T_{1/2,2}(9)=0$; *i. e.* $1+\frac{1}{2}\times 8-\frac{1}{8}\times 8^2=-9^{1/2}$.

(11) $T_{3/2,2}(1/9)=0$.

(12) $1-\frac{1}{2}(4-1)=-4^{-1/2}$.

II. Al-Bîrûnî's Computation of the Value of π

By Carl Schoy, Frankfurt am Main

Al-Bîrûnî,[1] who wrote at Ghazna *c.* 1000, was perhaps the first of the Arab writers to find the value of π to any very high degree of accuracy; or rather, in his case, the reciprocal of this value. In this he anticipated the work of al-Kashî[2] by about four centuries, although the value found by each was the reciprocal of that found by the other.

Al-Bîrûnî's treatment of the subject is found in the work *al-Qânûn al-Mas'ûdî*, Book III, Chapter V, and is as follows:[3]

It is a simple matter to express by a single number the ratio of the diameter of a circle to the circumference. Indeed, in making the attempt, altering the problem somewhat, the only major change refers to the procedure and improvement in the graduation of the circle, the extent depending upon the amount by which scientists differ.[4] The circumference consists of 360 parts, each part being divided sexagesimally. The origin (ašl) of this usage is to be found in the fact that the number 360 is the mean between the number of days of a sun year and a moon year, although there is no confirmation of this

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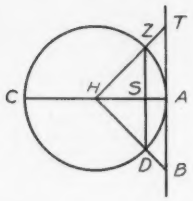
[1] Mohammed ibn Aḥmed, Abû'l Rîḥan (or Raiḥân), al-Bîrûnî (or al-Bêrûnî), a native of Khwarezm, *c.* 973–1038. See Smith, *History of Mathematics*, I, 285.

[2] Jemshîd ibn Mes'ûd ibn Maḥmud, Ġiyâṯ ed-dîn al-Kashî. He died *c.* 1436. See Smith, *loc. cit.*, I, 289, 290.

[3] Translated by David Eugene Smith from Dr. Schoy's manuscript.

[4] Cf. *Isis*, XIV, 371.



hypothesis.[1] The circumference of a circle bears a fixed ratio to its diameter, and if the number of the circumference be placed in proportion to that of the diameter there results the approximate number of this ratio.

On the figure there should be placed the necessary symbols. Through A, the extremity of the diameter AHC, we draw the perpendicular AT and connect (the center) H and Z,[2] producing it to cut the perpendicular at T. Because SZ is half the side of a 180-gon, which subtends two parts (degrees) of the 360 of the circumference, 2. SZ is the chord of 2°, the value of which is $2^{p}\ 5^{I}\ 39^{II}\ 43^{III}\ 36^{IV}$, the circumference being (approximately) made up of 180 such chords. The sum of all these chords is $6^{p}\ 16^{I}\ 59^{II}\ 10^{III}\ 48^{IV}$. Taking the diameter as 2, we have

$$\frac{\text{Diameter}}{6^{p}16^{I}59^{II}10^{III}48^{IV}} = \frac{1}{3^{p}8^{I}29^{II}35^{III}24^{IV}}. \tag{1}$$

But the circumference of the circle exceeds the sum of all these sides of the (regular) inscribed 180-gon.

Furthermore, since we have the proportion $TA:AH = ZS:SH$ we have $TA = 0^{p}\ 1^{I}\ 2^{II}\ 50^{III}\ 19^{IV}\ 43^{V}$; and since $TB = 2TA$, we have $TB = 0^{p}\ 2^{I}\ 5^{II}\ 40^{III}\ 39^{IV}\ 26^{V}$, and the number representing the perimeter of the circumscribed polygon is $6^{p}\ 17^{I}\ 1^{II}\ 58^{III}\ 19^{IV}$.

Hence we have

$$\frac{\text{Diameter}}{6^{p}17^{I}1^{II}58^{III}19^{IV}} = \frac{1}{3^{p}8^{I}30^{II}59^{III}10^{IV}}. \tag{2}$$

But the circumference of the circle is less than the sum of the sides of the [regular] circumscribed 180-gon, whence

$$\frac{1}{3^{p}8^{I}29^{II}35^{III}24^{IV}} > \frac{\text{Diameter}}{\text{Circumference}} > \frac{1}{3^{p}8^{I}30^{II}59^{III}10^{IV}}.$$

Therefore the circumference lies between the two values $3^{p}\ 8^{I}\ 29^{II}\ 35^{III}\ 24^{IV}$ and $3^{p}\ 8^{I}\ 30^{II}\ 59^{III}\ 10^{IV}$. We now make the same assumption that Ptolemy makes in Book VI of his Almagest, taking the arithmetic mean between the two, whence we have

$$\frac{\text{Diameter}}{\text{Circumference}} = \frac{1}{3^{p}8^{I}30^{II}17^{III}16^{IV}46^{V}30^{IV}}. \tag{3}$$

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[1] A very original idea of al-Bīrūnī's, but purely conjectural.

[2] As in the original. He seems to have taken AHZ as 1° and then to have constructed the chord of 2° the value of which he knew. See this MONTHLY, (*1926*, 95–96).

Since the value of that part of the denominator which follows 3^P is approximately 1/7, equation (3) may be written[1]

$$\frac{\text{Diameter}}{\text{Circumference}} = \frac{5184000}{1628681471}. \tag{4}$$

If we take the circumference as 360^P, the diameter is 114^P, and the remaining fraction is $954312306/1628681471 \cdot 360$, so that

$$\frac{\text{Diameter}}{\text{Circumference}} = \frac{114}{360} + \frac{954312306}{1628681471 \cdot 360}.[2] \tag{5}$$

III. New Light on Babylonian Mathematics

By L. C. Karpinski, University of Michigan

In his recent *Babylonia und Assyrien*,[3] Bruno Meissner, includes a summary of the natural and exact sciences as known to the Babylonians. This presents considerable material in the mathematical sciences not yet found in works on the history of mathematics although the material has been extant for some years in monographs and journals devoted to the history of the Orient.

The greatest interest attaches to two methods of approximation to find the diagonal of a rectangle in terms of the sides. The one method corresponds directly to the formula $\sqrt{x^2+a^2}=x+(a^2/2x)$ while the other method gives for $\sqrt{x^2+a^2}$ the value $x+2a^2x/3600$. Both methods are found on tablets

[1] The reciprocal value of the right side of (3) is 3.141742, and that of the right side of (4) is 3.141746, so that one agrees very well with the other. The approximation for π thus found by al-Bīrūnī is therefore somewhat larger than the value 3.14166, already known to the Greeks and the Hindus. At any rate the work in this chapter shows that the Moslem scientists had their own peculiar method of approach to the calculation of this ratio. The above plan shows clearly the line of approach and is much better than the one which Ibn al-Haitam (965–1039) had given in his work on the quadrature of the circle. (Compare H. Suter, "Die Kreisquadratur des Ibn al-Haiṭam," *Zeitschrift für Mathematik und Physik*, histor.-literar. Abteilung, 1899, pp. 33–47.)

Of the values for chord 2° and chord 1° calculated by al-Bīrūnī, he selected the worst (it being too large) for the determination of π. Similarly in his preceding computation, where he gives $AT=\tan 1°$, although it is greater than the values of tan 1° as given in his "table of shadows" (tangent table). (*Orient. MS.*, Okt. 275 Berlin, p 78, where $\tan 1° = 0^P\ 1^I\ 2^{II}\ 50^{III}\ 17^{IV}$.) For a later computation, in al-Bīrūnī's style, of a very exact value of chord 2°, compare the value of Jamshīd, al-Kashī, already mentioned, namely, $0^\circ\ 2^I\ 5^{II}\ 39^{III}\ 26^{IV}\ 22^V\ 29^{VI}$ (See *Isis*, No. 14, p. 389; Smith, *History*, I, 290), and for tan 1° the value given in the "table of shadows" of Ulugh Beg (*Berliner Pers. MS.* 280, fol. 36, *v.*), namely, $0^P\ 1^I\ 2^{II}\ 50^{III}\ 17^{IV}\ 38^V$, whence $=3.141671$. On Ulugh Beg (1393–1449), see Smith, *History*, I, 289.

[2] It is easy to derive equation (5) if we increase (4) by 360, multiply 5184000 by 360, and divide, the result by 1628681471, after the quotient 114 discontinues, etc.

[3] Heidelberg, 1925, 2 vols. In *Kulturgeschichtliche Bibliothek*. The references are to chapter 21, in vol. II, pp. 380–418.

in the British Museum and the methods were published in 1916 by E. F. Weidner;[1] both are applied to the rectangle of sides 40 and 10, giving 41:15, *i. e.*, $41\frac{15}{60}$, 42:13:20, or $42\frac{13}{60}\ \frac{20}{60^2}$, respectively. The material dates from *c.* 2000 B.C. The Pythagorean theorem is not mentioned, yet it is implicitly involved in the first method mentioned, if not in both.

Another interesting series of problems concerns the measurement of the area of irregular fields by subdividing the fields into a number of rectangles and triangles. The method is not materially different from modern methods with such areas.

Recent discoveries in the history of astronomy make it quite certain that Hipparchus drew considerable material from the Babylonian scientists. The above material, and equally the recently discovered Egyptian papyrus[2], show the dependence of early Greek mathematics (notably work on rectangles) upon Egyptian and Babylonian mathematics.

IV. Napier's Rods in China

By Père Louis Vanhée, S. J. Brussels

Professor Smith's statement in his *History of Mathematics* (vol. II, p. 203)' "The Napier Rods found their way unto China at least as early as the beginning of the 18th century," and Mr. David Chin-te Cheng's recent article in the Monthly (*1925*, 492), both open up a very interesting question as to the precise date of their introduction. Fortunately we have sources at hand which give us this information to a high degree of accuracy, but they seem not to have found their way heretofore into western literature relating to the history of mathematics.

The Jesuit fathers, Johann Adam Schall von Bell (1591–1666), a native of Cologne, and Giacomo Rho (1593–1638), a native of Milan,[3] held office in the Peking Astronomical Board near the close of the Ming Dynasty (1368–1644) and were the first to introduce this device into China. Since their time it has been known under the name *ch'ow suan* (*Computing by Rods*) and has been highly esteemed by Chinese astronomers and mathematicians. It is discussed by Mei Wen-ting (1633–1721),[4] the ablest Chinese mathematician of his time and the leading writer upon the history of the mathematical sciences of his people. His memoirs on the subject appeared in his collected works in sixty sections (the *Li-suan-ch'uan-shu*, published by Wei Li-t'ung

[1] "Die Berechnung rechtwinkliger Dreiecke bei den Akkadern um 2000 v. Chr.", *Orientalische Literaturzeitung*, vol. 19, columns 257–263.

[2] First published by B. Touraeff in *Ancient Egypt*, 1917, and later by the writer in *Science*, vol. 57, pp. 528-529.

[3] Smith, *History of Mathematics*, vol. I, p. 436.

[4] *Ibid.*, vol. I, p. 436; vol. II, p. 170.

in 1725). Unfortunately the term *ch'ow*, which is used to mean the Napier Rods, is the same one that had long been used to mean "counters," "computing sticks," and similar devices, and hence great care is necessary in order to avoid confusion as to its precise meaning. In Mr. Cheng's article, for example, the term "computing rods" is used in such a way as to add to the confusion and to obscure the real meaning of the expression.

In order to clarify the situation it should be observed that, according to the works of Rho, Schall von Bell, and Mei Wen-ting, who really set the standard of usage, the Napier Rods should be called *ch'ow*, and all the other similar instruments should be spoken of as *ts'e* (*computing sticks*, or *counters*). Mr. Cheng states (p. 494) that the use of "computing rods" began about the thirteenth century, and he gives a number of illustrations showing rods with numerals carved upon them. Now according to Mei Wen-ting[1] there were no numerals on the ancient Chinese counters, and he distinctly states that those having numerals were purely a European invention. It follows that whatever these old counters may have been, they were not of the type described by Mr. Cheng and certain other modern Chinese scholars. The rods with numerals upon them are, according to all reliable historical evidence, of European origin. Indeed, I have personally made a careful and extended analysis of Schall von Bell's astronomical works as published in Peking, this analysis appearing in the well-known Sinological monthly, *T'oung-pao* (Paris). In this it is clearly shown that the introduction of the rods is due to him and Rho as already stated.

Chinese mathematicians always make use of a loose terminology. For example, in 1744 Tai Chen published a work entitled *Ts'e-suan*, *ts'e* being merely another name for "counter" or "rod," and the work refers to Napier's rods, although this would hardly be inferred from the title. On the other hand, Ch'in K'iĕ, in his *Suan-fa-ta-ch'eng* (*Mathematical Compendium*) of 1843, a work in twenty-one sections, mentions the *suan pan* as the most convenient device for calculating, and the *ch'ow suan* (meaning the Napier Rods) as the next in order of merit. In this usage he followed an earlier compiler, *Fang Chung-t'ung*, who described the rods in his *Shu-tu-yen* (*Mathematical Summary*), a work written in 1687 but, owing to political reasons, not printed until 1721. In this he includes material from Matteo Ricci's translation of Euclid[2] and from an earlier treatise on the abacus, a section on arithmetic, one on the sector from Ricci's *T'ung-wen-suan-che*, and other material from the Jesuit Collection known as the *Sin-fa suan shu*, and other early treatises.

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[1] *Ku-suan-k'i-k'ao*. See Smith, vol. II, p. 170.

[2] Smith, *History*, vol. I, p. 303.

Mr. Cheng's article is valuable as showing the Chinese modification of the Napier Rods, and is very fair in its statement that the latter may have been due to the Jesuit fathers. What we would here emphasize is that this is the ca ', and we would call attention to the fact that the booklet by Fathers Schall von Bell and Rho is available in the large collection of works edited at Peking under the direction of Minister Siu.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

An Introduction to Mathematical Probability. By J. L. COOLIDGE. New York, Oxford University Press, American Branch, 1925. viii+216 pp. Price $5.00.

Calcul des Probabilités. By PAUL LEVY. Paris, Gauthier-Villars et Cie, 1925. viii+350 pages. Price 40 fr.

The publication by a firm with English and American connections of a text in English on mathematical probability written by an able and clear mathematician is a noteworthy and most welcome occurrence. Regardless of the requirement, laid down by most graduate schools, of a reading knowledge of French or German, it is nevertheless true that many American teachers and students are not willing and ready to delve in the existing treatises in those languages, particularly in a subject so full of pitfalls and often so difficult of exposition as the theory of probability. The book by Keynes has been sufficiently well criticized[1] to make it evident that it did not fill the need for such a text. It may be merely pointed out here that it was written, to quote Keynes' own words, "as a Fellowship Dissertation"; it can scarcely be styled a mathematical treatise, even though it may continue to be praised as a so-called philosophical treatment. One may well contrast the latter book with the very excellent but brief article "On Probability" wherein Charles Jordan[2] of the University of Budapest distinguishes sharply and accurately mathematical, empirical and philosophical probability, sketches concisely the leading theorems (Bayes, Bernoulli, Inversion, Rule of Succession), and discusses the mutual bearings of these without indulging in a wealth of "philosophical" discussion which brings about only a befogging of the ideas.

Coolidge's book may be divided roughly into the theory of probability, contained in Chapters I–VI, and its applications in Chapters VII–XI. After an introductory chapter in which he discusses the scope and meaning of mathematical probability and adopts three definitions suited to three chief

[1] Review by E. B. Wilson, *Bulletin A. M. S.*, vol. 29, 1923, p. 319.

[2] *Proc. Physico-Math. Soc. of Japan*, 3rd ser., vol. 7, no. 5, 1925, pp. 96–109.

conditions, he treats the elementary principles (total and compound probability, expectation, risk), Bernoulli's theorem, mean value and dispersion, geometric probability, and probability of causes. Only a thoughtful and repeated reading of the book will show how clearly a topic such as Bernoulli's theorem is presented and is related in its logical bearing to the first definitions and is safeguarded in its practical application to problems in chance. Equally important is it that our students have in Chapter VI a discriminating statement and discussion of Bayes' principle, both for a finite number of causes and for continuous probability, both for the probability of an operative cause and as applied to future events.

The later chapters treat errors of observation, errors in many variables, indirect observations, the statistical theory of gases, and life insurance so far as this involves considerations of probability. In determining "the best value" among a set of discordant measures, Coolidge lists carefully the postulates leading to the weighted arithmetic mean, preferring Schiaparelli's postulate of a singly differentiable continuous function of the measures to the Schimmack postulate sometimes used. Further assumptions, advanced and supported as being natural and plausible, lead to a modification of Gauss' first proof of his law of error; the author wisely prefers to adopt a single perspicuous set of assumptions rather than to adopt a less plausible proof which might be more elegant from the axiomatic standpoint or to make an extended critique of various methods of historical importance. Coolidge incorporates here his development of the Gaussian law of error for any number of variables as given in the *Transactions of the American Mathematical Society* for 1923, with a reference to the error ellipse applied in artillery practice and the "Big Bertha" shots of 1918. A fuller account of the use of the correlation coefficient in statistics with some illustrative examples might well have been given in the treatment of that subject. The method of least squares and the theory of curve fitting are treated satisfactorily but concisely, a particularly clear explanation of the meaning of the "best representation" of a given function by a curve being given. Depending on Castelnuovo as the moving spirit and assembling as before a list of clean-cut assumptions, the author develops Maxwell's law of the distribution of velocities in a gas and the supplementary work concerning the probability that a gas shall be in a nearly normal state. The book closes with the placing of the principles of life insurance on their basis in the theory of probability, a description of commutation tables, and, at the end, tables containing the common logarithms of e^x and e^{-x}, and the probability integral.

The book is made more valuable for class use through well selected problems throughout, especially in the first two-thirds of the book.

Levy consciously makes the law of large numbers (Bernoulli's) the central feature of his treatise and the foundation of the calculus of probability, problems in games of chance being regarded merely as simple problems which enable one to grasp the real meaning of the later principles. The second outstanding feature of the book is that Levy bases his rigorous development of Gauss' law for accidental errors on the notion of characteristic function, a method which at least in considerable part goes back to Cauchy and 1853. Levy justifies this, as against the direct derivation of Gauss' law from the binomial formula, on the ground that when once the fundamental properties of the characteristic function are found, a large number of important consequences are obtained, adopting thus a unifying principle similar, say, to that which Hilbert uses in his organization of integral equations.

The book is divided into two main parts. The first part on the principles of the calculus of probability contains chapters on subjective probability, principles derived experimentally, objective value of probability, laws of probability (the law of Gauss and its relation to the law of large numbers), probability derived from experience (statistics), and a critique of the theory of expectation or risk. The second part on the mathematical theory of probability treats in successive chapters general notions about the laws of probability and the theory of sets, probable values, characteristic coefficients and functions (his basic chapter, including his proof of Gauss' law of error), the composition or combination of laws of probability, laws of variable probabilities, a more extended study of the law of large numbers, exceptional laws the theory of errors, and the kinetic theory of gases.

W. D. CAIRNS

Romance in Science. By BESSIE I. MILLER. Boston, The Stratford Co., 1924. 87 pages. Price $1.00

This very interesting little book contains a series of lectures delivered by the author in an elective course offered by the mathematical department of Rockford College. It first appeared in the college catalogue under the double heading, "The Romance of Science," or "The Grammar of Science," according to which you wish to get out of it. Officially it is known as "Some Literature of Science." But in the registrar's office it has always been known by the significant title *Browse*, and that is evidently the name favored by the author. We read:

"It is Browse which best describes the method of study advised in the course. Have you ever seen a sleek, biscuit-colored Jersey cow, hock deep in the lush grass of a meadow where it slopes down toward a transparent stream tinkling over brown stones in the shadow of large weeping-willow trees? It is a Jersey cow which browses. A Jersey crops only now and then, and between times gazes meditatively over the landscape."

That it is a very rich meadow growing an extraordinary number of different varieties of grasses in which this browsing is done is evident from the following

chapter headings. The scientific method, Law, The fourth dimension, The fourth dimension and non-Euclidean geometry, Einstein's theory of gravitation, Transformations, The human significance of mathematics.

The "browsing" is done not only in the field of mathematics, but also in astronomy, architecture, physics, biology, literature, philosophy, psychology, geology, chemistry, medicine, law, art, music, and religion. Like *The Education of Henry Adams* this book is delightfully informal, not to say unmethodical and contains many interesting and instructive bits of philosophy.

To an old college teacher of mathematics the most astonishing thing about this "*Browse*" course is that the prerequisite for it is only "Freshman mathematics or enrollment in that course with a minimum grade of B for the first semester. This secures some knowledge of analytical geometry, limits, and differentiation on the part of all, although it is a very elementary knowledge." That the lectures are delivered to college girls, some of whom do not even elect mathematics after the freshman year, also seems rather extraordinary.

One meaning of browse, as given in the dictionary, is: "To eat or nibble off, as the tender branches of trees, shrubs, etc." This is the kind of browsing preferred by most of the undergraduates I have met; they are very partial to those tender shoots from the tree of knowledge that are easy to masticate and digest. Like children they like to have the meal all dessert. But that is bad for the digestion. A little browsing in every lesson, in any subject, is a good wholesome dessert; that is what makes a subject interesting to the student. But it should be given in homeopathic doses and confined to topics within the range of the student's knowledge and mental capacity.

In the appendix is found a Reading List for "Browsing," comprising 63 books ranging all the way from *Jungle Peace* by Beebe, through the whole gamut of scientific knowledge, to *The Einstein Theory of Relativity*, by Lorentz.

W. A. GRANVILLE

A Brief Course in Analytic Geometry and the Elements of Curve Tracing. By W. B. FORD. New York, Henry Holt and Co., 1924. 288 pages. Price $2.40.

This text follows pretty closely the usual material presented in such a book. Starting with the definition and fundamental ideas, the author works through chapters on the line, circle, and the conics in the usual order, the transformation of coordinates, and polar coordinates. Then comes a chapter on other well known curves and one on curve fitting, a subject which seldom is found in a book of this kind. There is also a very short study of solid geometry, limited as usual to the line, plane and quadric. The book closes with a few trigonometric formulas and tables, answers to the problems, and an index.

Only a reference is made to oblique coordinates and in many places the author hesitates to give much of what one might call theory. If he showed that the graph of an equation is unaltered by such operations as the transposition

of a term, or multiplication or division by a constant different from zero, he would not have to give such definite instructions about the reduction of the general linear equation to any of the standard forms. It seems unfortunate to give the student the impression that he is not qualified to study a proof of the theorem that a first degree equation in two variables must graph into a straight line. The theorem on the classification of the graphs of quadratic equations is stated without reference to the exceptional cases. The degenerate cases are picked up in the next article, but no reference is made to the fact that all the solutions of an equation may be imaginary.

The author's choice of material in the chapters on the conics is very good, though some will wish that he had included a study of the diameter as a locus of middle points. The figures are very good and the general form is very pleasing. C. R. MACINNES

Analytic Geometry. By R. W. BRINK. New York, The Century Co., 1924. 274 pages. Price $2.25.

This is intended to be a text book for the usual course in the subject. It aims to be a book on the general subject of analytic geometry rather than one on the conic sections, as so many have been. The author opens with a short chapter devoted to things involving the coordinates of isolated points, such as the distance between two points and the point dividing a line segment in a given ratio. One wonders why it did not include also the slope of the line joining two points and the area of a triangle, derived by the trapezoid method, a method which seems to most beginners less cumbersome than the one involving the base and altitude. This is followed by a chapter on the straight line and by one giving a general discussion of the graphs of equations, including the ideas of symmetry, values of one coordinate for which the other becomes imaginary or infinite, and the degenerate cases. Then follow chapters on locus problems, transformation of coordinates, polar coordinates, and tangents. After these general ideas are disposed of, the author brings in two substantial chapters, the first on the circle and the next on the other conics. It seems strange that the existence of a second focus and directrix in the ellipse and hyperbola should not be mentioned. A chapter on some well known curves finishes plane geometry and a short study of solid geometry closes the book. This includes the line, plane and quadrics, except the cone.

An appendix gives a collection of formulas of trigonometry and algebra. It also gives the fundamentals of two and three row determinants, which are used throughout the book. The author has gathered a great many problems of fairly varied type, some with answers and many more without. Oblique coordinates are not even mentioned. The figures are very good and the detail work has been well done. Problem 2, page 156, might have been more carefully stated. C. R. MACINNES

Statistical Analysis. By E. E. DAY. New York, MacMillan Co., 1925. xxvii+459 pages. Price $4.00

While this book does not "enter upon the more refined mathematical phases of statistical method" and therefore calls for little comment here, it is of importance that no serious objection can be found to the statement that "nothing in the book should prove inconsistent with the findings of the most advanced mathematical statistics." Writers of such books and teachers of corresponding courses are not always as sympathetic toward the demands of the mathematician for rigor and it is a distinct pleasure to recommend this book for its sound doctrine and logic. A more extended title of "The Logic of Statistical Analysis" is suggested by the author.

In addition to the traditional treatment of averages, correlation, etc., there is, at the beginning, a thorough discussion and treatment of the various terms used in the tabulation, classification and analysis of observed data. The latter part is devoted to the treatment of time series and of index numbers. The treatment throughout is careful and thorough. C. H. FORSYTH

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

American Journal of Mathematics, volume 48, no. 1, January, 1926: "Maps of Twelve Countries with Five Sides with a Group of Order 120 Containing an Ikosahedral Subgroup" by H. R. Brahana and A. B. Coble, 1–20; "On Surfaces and Curves which are Invariant under Involutory Cremona Transformations" by A. Emch, 21–44; "The Invariant System of Two Associated Bilinear Connexes" by O. E. Glenn, 45–56; "Arithmetics of Generalized Quaternion Algebras" by C. G. Latimer, 57–66; "Conditions under which One of Two given Closed Linear Point Sets may be Thrown into the Other One by a Continuous Transformation of a Plane into Itself" by R. L. Moore, 67–72.

Annals of Mathematics, second series, volume 27, no. 2, December, 1925: "Notes on the Ampère-Cauchy derived function" by H. L. Smith, 69–72; "Note on the existence of analytic solutions of non-homogeneous linear q-difference equations, ordinary and partial" by C. R. Adams, 73–83; "Incidence and parallelism in biaffine geometry" by A. A. Bennett, 84–86; "New proofs of the simplicity of every alternating group whose degree is not four" by G. A. Miller, 87–90; "Symmetric tensors of the second order whose covariant derivatives vanish" by H. Levy, 91–98; "Generalizations of the eight square and similar identities" by E. T. Bell, 99–104; "On a geometrical theory of continuous groups. I. Families of path-curves of continuous one-parameter groups of the plane" by B. de Kerékjártó, 105–117; "Group velocity and the propagation of disturbances in dispersive media" by F. Wood, 118–128; "On an expression of a minor of order two of the Mth compound of a determinant A in terms of minors of A of order higher than M" by W. H. Metzler, 129–132; "The nullity of a matrix relative to a field" by C. C. MacDuffee, 133–139; "Determination of the ternary collineation groups whose coefficients lie in the $GF(2)$" by R. W. Hartley, 140–158; "On the oscillation of harmonic functions" by F. W. Perkins, 159–170.

Journal of Mathematics and Physics, Massachusetts Institute of Technology, volume 5, no. 2, February, 1926; New Formulation of the Laws of Quantization of Periodic and Aperiodic Phenomena"

by Max Born and Norbert Wiener, 84–98; "The Harmonic Analysis of Irregular Motion" by Norbert Wiener, 99–121; "A Method of Solving Power Networks by Means of Conjugate Vectors" by W. L Lyon and F. L. Hitchcock, 122–125; "Note on a Method of Evaluating the Complex Roots of a Quartic Equation" by Y. H. Ku, 126–128.

Mathematische Zeitschrift, volume 24, no. 4, February, 1926: "Note on the location of the roots of a polynomial" by J. L. Walsh, 733–742.

UNDERGRADUATE MATHEMATICS CLUBS

EDITED BY H. J. ETTLINGER, 3110 Harris Park Ave., Austin, Texas

CLUB ACTIVITIES

PI MU EPSILON, Hunter College.

The following play was put on by the Hunter College chapter of Pi Mu Epsilon at the American Mathematical Society dinner, Jan. 1, 1926. The author requests that his name be withheld.

A RELATIVITY DRAMA

CAST OF CHARACTERS: HUMANITY, LIGHT, MICHELSON, MORLEY, MILLER, EINSTEIN, FOUR ASTRONOMERS, FIVE TOOLS.

Light, a young woman, is beloved of all on account of her many virtues and extreme usefulness. She has been accused by the astronomers of an affair with the Sun and, at their behest, has been summoned into court to be tried before Humanity. The scene opens with Humanity sitting as judge, the astronomers as plaintiffs on the right. To the left stand certain persons who describe themselves in the action. To the left forward, stands Light herself in the prisoner's box.

HUMANITY: You astronomers, here you are with a charge against fair Light. Let me warn you that it must be a clear case and well proved, otherwise, you will be eternally discredited in our eyes. Stand out one at a time! Who are you?

FIRST ASTRONOMER: I, noble Humanity, am a noted professor from ....

SECOND ASTRONOMER: I, your Honor, am that most learned and far-seeing astronomer from ....

THIRD ASTRONOMER: I, your Honor, am that great astronomer who writes popular astronomy for the ....

FOURTH ASTRONOMER: I am from the famous observatory ....

HUMANITY: State your case, first Astronomer.

FIRST ASTRONOMER: Your Honor, we astronomers are great detectives. None can see as far as we. We look, no detail escapes us.

ALL (*slowly*): Look, look, look.

FIRST ASTRONOMER: We measure as no other can measure.

ALL: Measure, measure, measure.

FIRST ASTRONOMER: We have turned our attention on Light. She is young and good to look upon and as such, as your Honor well knows, should constantly go in a straight line. Such a one had we thought her. But (ALL: Sh! sh! sh!) we were mistaken. We have seen her pass by the Sun. We have focused our instruments upon her. Verily, to see her at eclipses, we have journeyed to the South Seas, to barbarous Mexico, to desert sands and to arctic New Haven. Each time she passes the Sun, she departs from the straight line of rectitude.

ALL: Here are the instruments, the charts, the pictures.

FIRST ASTRONOMER: Behold the telescope!

SECOND ASTRONOMER: Behold the blue prints!

THIRD ASTRONOMER: Behold the pictures!

FOURTH ASTRONOMER: Behold the bent line!

HUMANITY: Ah! Ah, a grave charge indeed.

ALL: Grave, very grave.

HUMANITY: Light, what have you to say for yourself?

LIGHT (*timidly*): Your Honor, if I did turn from the straight line a teeny, weeny bit, it was all due to Gravity, the all powerful. What could a poor girl do? It was not my fault, not a bit.

FIRST ASTRONOMER: You see she admits it. A clear case.

SECOND ASTRONOMER: And Gravity, how ridiculous! I know all about Gravity.

ASTRONOMERS: And I, and I.

HUMANITY: Stand out and tell about Gravity, you from ....

ASTRONOMER: Gravity, your Honor, is a very strong demon who lies in wait to grab all kinds of men. It grabs and pulls and pulls but,—Gravity never possibly touched a woman. Light herself knows that Gravity has nothing whatever to do with her nor she with Gravity.

FIRST ASTRONOMER: Right, right, Gravity never influenced Light.

ALL: Impossible, impossible.

HUMAINTY: Astronomers, where did you learn about Gravity?

FIRST ASTRONOMER: Newton told us.

SECOND ASTRONOMER: Yes, the great Newton.

ALL: Newton, Newton.

HUMANITY: But Newton has been dead over two hundred years.

FIRST ASTRONOMER: No matter.

SECOND ASTRONOMER: Newton knew.

HUMANITY (*dubiously*): Where did Newton learn?

ASTRONOMER: He saw an apple.

ALL (*slowly*): Apple, apple, apple.

HUMANITY: I do not understand but we shall proceed. Are there witnesses for Light? (*Michelson steps out*).

HUMANITY: Your name, please.

MICHELSON: I am Michelson of Chicago.

HUMANITY: Do you know Light?

MICHELSON: All my life I have loved her. All my life I have studied her.

HUMANITY: Tell us of Light.

MICHELSON: Light, oh learned Judge, is a young woman of perfect honor. I tell what she does by the fringes of her hair.

HUMANITY: How so? But never mind, proceed.

MICHELSON: She goes at the same speed going toward the Sun or coming from it, whether she goes with you or across your path. It matters not whence she comes nor whither she is going.

HUMANITY: You have measured her speed?

MICHELSON: Yea, verily. (*Morley steps out*)

MORLEY: Your Honor, I am Morley, I agree with Michelson. (*Miller jumps up*)

MILLER (*loudly*): Your Honor, I disagree. The ether . . .

HUMANITY: Who are you?

MILLER: I am Miller of the Case School.

HUMANITY: Sit down. Michelson, what of Gravity?

MICHELSON: I have not studied him.

HUMANITY: Are there other witnesses for Light? (*Einstein steps out and remains standing without speaking*)

HUMANITY: Your name?

EINSTEIN: I am Einstein.

HUMANITY: What know you of this case?

EINSTEIN: I am a mathematician.

HUMANITY: Ah, a worthless calling.

EINSTEIN: I think not, your Honor, and I hope by this case to show you the contrary.

HUMANITY: The truth is, no one knows you. Mathematicians do not often put themselves before Humanity. How am I to know it? Proceed.

EINSTEIN: I have here my tools. Stand forth and announce yourselves.

FIRST TOOL: I am Noneuclidean Geometry. My parents Riemann and Lobatchewsky were the greatest of men.

SECOND TOOL I am Space-time. My father, Minkowski stands second to none.

THIRD TOOL: I am her sister, Higher Dimensional Geometry. No one boasts more honorable descent than I.

FOURTH TOOL: I am Tensors, young but as good a woman as any.

EINSTEIN: I have other tools that are not here but the greatest is here. Now, you the chief of all my tools, my right hand, step out.

FIFTH TOOL: I am Mathematical Analysis.

HUMANITY: And now what of the case?

EINSTEIN: We have worked with unerring accuracy. These my tools never fail me. I, too, love Light. I have striven and have vindicated her name. There is no Demon Gravity. Poor Light did not understand. She could not help her actions which were due to the position she was in, solely. The course of correct behavior was to bend the path as she bent hers; to go not in a straight line, but in a hyperbola, and this is not what astronomers think it is, for in my opinion space is non-Euclidean. So would the astronomers, perforce. They, poor old fogies, wished the modern Woman, Light, to go according to their old-fashioned notions, the same as with Newton over two hundred years ago. But Light, the girl personified, goes according to where she is and swiftly.

HUMANITY: The case is dismissed. Light, I congratulate you. Einstein, you and Michelson escort Light from the court.

All go out. Michelson, Einstein, Light, arm in arm.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF NEBRASKA, Lincoln, Nebraska

[*1924*, 146]

The program for the year 1924–1925 was the following:

November 6, 1924. "Magic squares" by Dr. A. L. Candy.
December 3. Social meeting.
February 25, 1925. "Construction of triangles" by Mr. Ogden.
March 25. "Lenses and cameras" by Mr. Russell.
April 20. "Relation of mathematics to astronomy" by Professor Sweezy.
May 20. Annual picnic.

(Report by Irma E. Wiedeman, secretary)

THE EUCLIDEAN CIRCLE OF THE UNIVERSITY OF INDIANA, Bloomington, Indiana

[*1924*, 495]

The program for the year 1924–1925 was the following:

November 10, 1924. "The continuity of the number system and the creation of rational numbers based on the Dedekind postulate" by Myrtle M. Comer. "Archimedes" by T. H. Rawles.
November 17. "The development and application of mathematics" by Professor S. C. Davisson.
January 19, 1925. "Ciphers" by Professor J. S. Galland.
March 30. "Relativity and the relation of philosophy to science" by Chas. Kasper.
April 27. "Asteroids" by N. H. Long.
May 18. "The cosine theorem as it is developed from the analytical standpoint" by Professor D. A. Rothrock.

(Report by Professor H. T. Davis)

PI MU EPSILON, UNIVERSITY OF PENNSYLVANIA, Philadelphia, Pa.

[*1922*, 148]

For the first time since the University of Pennsylvania chapter was organized, the undergraduate members of the fraternity have been taking an active part in the lectures and mathematical discussions offered by members and friends. It is hoped that this active participation of the younger element in

the affairs of the fraternity will serve as an impetus towards increased attendance at the meetings of the fraternity and towards increased interest being shown by the student body of the school.

The folowing papers were read during the year 1924–1925:

1. "Mathematical fallacies" by Miss S. M. Wolfe.
2. "Theory of numbers" by Dr. E. E. Witmer.
3. "Introduction to the calculus of variations" by Dr. J. D. Eshleman.
4. "The duplication of the cube" by Miss R. H. Stauffer.
5. "The Einstein theory" by Mr. L. Zippin.
6. "Stereographic projection" by Mr. Chadwick.
7. "Some regular polyhedra" by Miss I. Shields.
8. "English mathematics and mathematicians" by Dr. Wilson of Haverford College.

(Report by Miss Ruth H. Stauffer, secretary)

The Mathematics Club of Albion College, Albion, Mich.

The program of the Mathematics Club of Albion College for the year 1924–25 was the following:

October 14, 1924. Election of officers for the first semester.

President, William Wylie, '25;
Vice-president, Irene Bauer, '25;
Secretary-treasurer, Esther Richard, '26;
Member of the Program Committee, Omar Bartow, '25.

October 21. "Unrealized possibilities in teaching mathematics" by William Wylie, '25.

November 18. Roll call responded to by giving a trigonometric formula. "Early mathematics in the United States" by Harmon Camburn, '25. "Early mathematics in the United States by Arthur Dewey, '26. Critic's report by Omar Bartow, '25.

December 2. Roll call responded to by giving a geometrical construction. "Ruler and compasses construction" by Omar Bartow, '25. Critic's report by William Wylie, '25.

January 6, 1925. "How to make geometry interesting" by Esther Richard, '26. "Some problems to be met by a mathematics teacher" by Keith Friend, '25. Critic's report by Irene Bauer, '25.

Election of officers for the second semester.

President, George Price, '26;
Vice-president, Omar Bartow, '25;
Secretary-treasurer, Rupert Cortright, '26;
Member of the Program Committee, Keith Friend, '25.

February 3. Roll call responded to by facts from the lives from English mathematicians. "The planimeter" by Lester Smith, '26. "The slide rule" by Mr. Sears. Critic's report by Omar Bartow, '25.

March 3. Roll call responded to by facts from the lives of German mathematicians. "Magic squares" by Irene Bauer, '25. "Algebraic magic squares" by George Price, '26. Critic's report by Lester Smith, '26.

April 21. "Method of extracting the square root" by Arthur Dewey, '26. "Method of extracting any root" by Harmon Camburn, '25. "The teaching of logarithms" by Frank Sanders, '26. Critic's report by Irene Bauer, '25.

May 5. Debate: Resolved, that two years of mathematics should be required in the high school,—that two years to consist of one year of algebra and one year of geometry.

Affirmative:	Negative:
Esther Richard, '26	Keith Friend, '25
William Wylie, '25	Rupert Cortright, '26

May 19. Social meeting.

(Report by Rupert Cortright, secretary)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3187. Proposed by Frank Morley, Johns Hopkins University.

Taking six points, 1, 2, 3, 4, 5, 6, on a conic, let the areas of the triangles 612, 123, $\cdots$ be denoted by $a_1, a_2, \cdots, a_6$. Prove that the area, A, of the hexagon, formed by the points as ordered, is given by

$$(A-a_1-a_3-a_5)a_2a_4a_6=(A-a_2-a_4-a_6)a_1a_3a_5.$$

3188. Proposed by B. F. Finkel, Drury College.

Find the equation of the curve whose radius of curvature at any point of the curve is n times the radius vector to the same point.

3189. Proposed by C. G. Latimer, Tulane University.

Find integers, h, k_i, c_i, n, where $|k_i|<h$, such that

$$\log(x+h)=\sum_{i=1}^{n}c_i.\log(x+k_i)+\sum_{i=1}^{\infty}a_iz_i, \qquad (i=1,2,\cdots),$$

where the a_i are constants and $z=b/f(x)$ where b is a constant and $f(x)$ is a polynomial of degree $\geqq 6$.

Borda's and Haro's series are similar to this except that they involve polynomials of the third and fourth degrees respectively.

3190. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Two given lines x, y are met in the points P, Q by a circle passing through their common point and through another fixed point in the plane. Find the locus of the point of intersection of the lines projecting P, Q upon y, x, respectively.

3191. Proposed by A. A. Bennett, Lehigh University.

Prove that each positive integer less than 2×10^9, has the property of containing an odd prime factor or else of being expressible as a product (with distinct factors) of the form $\Pi_i(p_i-1)$ where each p_i is a prime.

3192. Proposed by J. L. Riley, Ouachita College.

Find all pairs of numbers x and y such that $rx \equiv s \pmod{y}$ and $r'y \equiv s' \pmod{x}$; r, s, r', s', being given integers.

3193. Proposed by W. H. Rasche, Virginia Polytechnic Institute.

Prove that two coplanar, copolar triangles are homological; and, conversely, two coplanar, homological triangles are copolar.

NOTE.

(1) In the proof of the foregoing theorem, use only Euclidean geometry and the positive and negative signs to distinguish the internal division of a linear segment from its external division.

(2) It is of interest to note, too, that a similar theorem holds for spherical geometry, namely: Two spherical, Coplanar triangles are homological; and, conversely, two spherical homological triangles are coplanar, two spherical triangles ABC and A'B'C' being coplanar when the great circle arcs AA', BB', CC' are concurrent and two spherical triangle ABC and A'B'C' being homological when the three pairs of corresponding sides meet in three points which lie on the same great circle.

SOLUTIONS

Problem 2676 [1918, 75]. **Proposed by E. R. Smith, Iowa State College.**

Find the greatest term of the series

$$\frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}F(-r,\ -sq,\ sp-r+1,\ 1)$$

where s, r, sq and sp are positive integers, $r<s$, p and q are proper fractions such that $p+q=1$, and $F(-r,\ -qs,\ sp-r+1,\ 1)$ is a hypergeometric series. If s,r, $s-r$ are large, show that the greatest term is approximately equal to

$$\sqrt{\frac{s}{2\pi pqr(s-r)}}.$$

SOLUTION BY F. M. WEIDA, Lehigh University.

Let us consider the function

$$\phi=\frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}F(a,b,c,z) \tag{1}$$

$$\text{where } F(a,b,c,z)=1+\frac{a\cdot b}{1!c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+\cdots.$$

If in (1) we put $a=-r$, $b=q$, $c=sp-r+1$, $z=1$, it is readily found that

$$\phi=\frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}\left\{1+\frac{rqs}{ps-r+1}+\frac{r(r-1)}{2!}\cdot\frac{qs(qs-1)}{(ps-r+1)(ps-r+2)}+\cdots\right\}. \tag{2}$$

At this point, it may be of interest to note[1] that the successive terms of (2) give the chances, respectively, of getting r, $r-1$, $\cdots$, 0, black balls from a bag containing ps black and qs white balls when r balls are drawn from the bag.

Let us now write the general term of (2), namely,

$$\phi_x=\frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}\cdot\frac{r(r-1)\cdots(r-x+2)}{(x-1)!}\cdot\frac{qs(qs-1)\cdots(qs-x+2)}{(ps-r+1)\cdots(ps-r+x-1)}. \tag{3}$$

It is now our problem to find the value of x in (3) for which ϕ_x as given by (3) is a maximum. Now

$$\Delta y_x=y_{x+1}-y_x=y_x\left\{\frac{r-x+1}{x}\cdot\frac{qs-x+1}{ps-r+x}-1\right\}=y_x\left\{\frac{(r+1)(qs+1)-x(s+2)}{x(ps-r+x)}\right\} \tag{4}$$

for $p+q=1$.

Imposing the condition for a maximum, we find from (4) that

$$(r+1)(qs+1)-x(s+2)=0.$$

If $x<(r+1)(qs+1)/(s+2)$, then $\Delta y>0$; while if $x>(r+1)(qs+1)/(s+2)$, then $\Delta y<0$. Since

$$\lim_{s\to\infty}\left[\frac{(r+1)(qs+1)}{s+2}\right]=rq+q,$$

[1] See Elderton, *Frequency Curves and Correlation*, p. 37.

we find that, if r and s are sufficiently large, ϕ_x is a maximum when

$$x=rq. \tag{6}$$

Substituting the value of $x=rq$ in (3) and combining terms, we find that

$$\phi_{rq}=\frac{ps(ps-1)\cdots(ps-pr)}{s(s-1)\cdots(s-r+1)}\cdot\frac{r(r-1)\cdots(rp+2)qs(qs-1)\cdots(qs-qr+2)}{(rq-1)!}. \tag{7}$$

From (7) we see that we may also write that

$$\phi_{rq}=\frac{(ps)!\times(s-r)!\times(r)!\times(qs)!}{(ps-pr-1)!\times(s)!\times(r-rq+1)!\times(qs-rq+1)!\times(rq-1)!} \tag{8}$$

Let us now replace each factorial in (8) by Stirling's[1] approximation to the factorial, namely,

$$n!=n^n\times e^{-n}\times\sqrt{2\pi n}, \text{ and we have}$$

$$\phi_{rq}=\frac{(ps)^{ps+1/2}\times(s-r)^{s-r+1/2}\times r^{r+1/2}(qs)^{qs+1/2}}{\sqrt{2\pi}(ps-pr-1)^{ps-pr-1/2}\times s^{s+1/2}\times(rp+1)^{rp+3/2}\times(qs-qr+1)^{qs-qr+3/2}\times(rq-1)^{rq-1/2}} \tag{9}$$

If in (9), we divide $(ps-pr-1)$ into two factors of which $(s-r)^{ps-pr-1/2}$ is one of the two; and if we divide $(rp+1)$ into two factors of which $r^{rp+3/2}$ is one of the two; and if we divide $(qs-qr+1)$ into two factors of which $(s-r)^{qs-qr+3/2}$ is one of the two; and if we divide $(rq-1)$ into two factors of which $r^{rq-1/2}$ is one of the two; and if we then impose the condition that $r,s,\ s-r$ are large, it is easily seen after combining terms and simplifying that, approximately,

$$\phi_{rq}=\sqrt{\frac{s}{2\pi rpq(s-r)}} \tag{10}$$

and this is the result desired.

3123 [1925, 138]. **Proposed by B. F. Finkel, Drury College.**

A circular hole, radius r, in the bottom of a flat-bottomed water-tank is covered with a weightless spherical rubber shell, radius R. Water is then poured into the tank to the depth h. What is the ratio of R to r when the shell is just on the point of rising?

II. Solution by the Editors.

Let R be the radius of the spherical shell and r_2, the radius of the circular hole in the bottom of the tank. If water is poured into the tank, the only portion of the ball's surface upon which there is an upward pressure lies between the level of depth $\sqrt{R^2-r_2^2}$ and the bottom, and the portion upon which there is a downward pressure lies above the first level. To each zone of the sphere between this level and the bottom there is an equal zone above this level symmetrical to the first zone. Since the pressure on the lower zone is greater than that on the upper, the ball will rise if $h\leqq 2\sqrt{R^2-r_2^2}$. We have now two cases to consider: Case I. $2\sqrt{R^2-r_2^2}<h\leqq R+\sqrt{R^2-r_2^2}$; Case II. $h>R+\sqrt{R^2-r_2^2}$.

Case I. Let r_1 be the radius of the section of the surface of the water with the sphere, and x, the radius of a horizontal section between the two sections of radii r_1 and r_2. Then $h=\sqrt{R^2-r_2^2}+\sqrt{R^2-r_1^2}$. The force acting vertically downward upon the wetted part of the segment of radius r_2 is

$$F_1=2\pi wg\int_{r_1}^{r_2}\left[x\sqrt{R^2-r_1^2}-x\sqrt{R^2-x^2}\right]dx$$

$$=2\pi wg\left[\frac{1}{2}(r_2^2-r_1^2)\sqrt{R^2-r_1^2}+\frac{1}{3}(R^2-r_2^2)^{3/2}-\frac{1}{3}(R^2-r_1^2)^{3/2}\right].$$

[1] Whittaker and Robinson, *Calculus of Observations*, p. 140.

The buoyant force is the weight of the water which could fill the space between sphere and a cylinder of radius r_2,

$$F_2=-\frac{4\pi wg}{3}(R^2-r_2^2)^{3/2}.$$

If $F_1+F_2=0$, we have after certain cancellations $h=3\sqrt{R^2-r_2^2}$.

If $r_1=0$, $h=R+\sqrt{R^2-r_2^2}=3\sqrt{R^2-r_2^2}$. Hence $r_2/R=\sqrt{3/2}$, $h=3R/2$.

Case II. $h>R+\sqrt{R^2-r_2^2}$, Here

$$F_1=2\pi wg\int_0^{r_2}[h-\sqrt{R^2-r_2^2}-\sqrt{R^2-x^2}\,]xdx,$$

$$=2\pi wg\left[\frac{r_2^2}{2}(h-\sqrt{R^2-r_2^2})+\frac{1}{3}(R^2-r_2^2)^{3/2}-\frac{R^3}{3}\right].$$

Hence if $F_1+F_2=0$, we have $(2R^2+r_2^2)\sqrt{R^2-r_2^2}=3hr_2^2-2R^3$; and setting $r_2/R=k$, we have

$$(2+k^2)\sqrt{1-k^2}=\frac{3hk^2}{R}-2,$$

an equation of the sixth degree in k when freed of radicals.

Also solved by H. S. UHLER whose solution was similar to that of E. M. BERRY (*1926*, 50).

3137 [1925, 261]. **Proposed by Harry Langman, New York City.**

Show how to draw, using straight-edge only, a tangent to the circumference (or an arc) of a circle at a given point, without making use of Pascal's hexagon theorem.

SOLUTION BY NINA M. ALDERTON, Mills College.

Through P, the given point on the arc C of the circle, draw a straight line cutting the arc again at A. Take any point P_1 on PA and draw through it the chord BD. Draw the lines AB,DP meeting at E, PB, DA meeting at F. Then the line EF is the polar of P_1. Now take a second point P_2 on PA and repeat this construction finding the polar MN of P_2. Let MN and EF meet in L, which must be the pole of PA. Then the line drawn through L and P is the tangent at P.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Associate Professor A. F. KOVARIK has been promoted to a full professorship of physics at Yale University.

Associate Professor LOUIS LINDSEY, of Syracuse University, has been promoted to a full professorship of applied mathematics.

Assistant Professor J. C. MOREHEAD has been promoted to an associate professorship at the Carnegie Institute of Technology.

At the University of Michigan the following appointments to assistant professorship are announced: G. Y. RAINICH, formerly lecturer and research fellow

at Johns Hopkins; R. L. WILDER, formerly assistant professor at Ohio State University; J. A. NYSWANDER, formerly assistant professor at Swarthmore College.

Dr. F. H. MURRAY, of Dalhousie University, has been promoted to an assistant professorship of mathematics.

Dr. J. R. MUSSELMAN, of John Hopkins University, has been promoted to an associate professorship of mathematics.

Miss MARGARET C. PACKER, of Hood College, has been promoted to an associate professorship of mathematics.

Assistant Professor L. P. SICELOFF, of Columbia University, has been promoted to an associate professorship of mathematics.

Professor J. D. TAMARKIN has been appointed assistant professor of mathematics at Dartmouth College.

Professor J. H. TANNER, head of the department of mathematics at Cornell University, has retired.

Assistant Professor J. S. TAYLOR, of the University of Pittsburgh, has been promoted to an associate professorship of mathematics.

Assistant Professor J. T. TRACEY, of Yale University, has been promoted to an associate professorship of mathematics.

Dr. M. S. VALLARTA has been promoted to an assistant professorship of physics at the Massachusetts Institute of Technology.

Associate Professor R. M. WINGER, of the University of Washington, has been promoted to a full professorship of mathematics.

Dr. EUPHEMIA R. WORTHINGTON, of the Southern Branch of the University of California, has been promoted to an assistant professorship of mathematics.

The following appointments to instructorships are announced:
Bryn Mawr College, Dr. ECHO D. PEPPER;
University of Michigan, Dr. F. W. KOKOMOOR; I. M. SHEFFER;
University of Rochester, Miss ROSE L. ANDERSON;
Rutgers University, Dr. C. M. HUBER and Mr. A. E. MEDER.

Professor R. D. BOHANNAN, who had just completed thirty-nine years as head of the department of mathematics at the Ohio State University, died suddenly June 20, 1926, at the age of seventy-one. His death was due to apoplexy.

Professor S. M. BARTON, of the University of the South, died January 5, 1926, at the age of sixty-six.

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.

Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.

The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7–8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Annapolis, Md., December 4.

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Colorado College, April, 1927.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

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THE MATHEMATICAL ASSOCIATION

The following forty-five persons and five institutions have been elected to membership in the Association, their applications having been duly certified.

To Individual Membership

ATKINSON, JULIA F., A.M. (California). Asst. Prof., Univ. of Arizona, Tucson, Ariz.

BAKER, FRANCES E., M.S. (Iowa). Instr., Tabor College, Tabor, Ia.

BASSLER, KATHARINE R., A.M. (Bryn Mawr). Teacher, Bryn Mawr School for Girls, Baltimore, Md.

BAUR, P.E., A.B. (Baldwin-Wallace). Asst. Prof., Math. and Drawing, Baldwin-Wallace Coll., Berea, Ohio.

BEATTY, H. M., A.M. (Ohio State). Asst. Prof., Ohio State Univ., Columbus, Ohio.

BETANCOURT DEL VALLE, R., Commercial Eng., (Inst. Polytech. of Seville), B.C.S. (Lincoln-Jefferson). Accountant, Plazuela Sugar Co., San Juan, P. R.

BIAS, J. H., A.B. (Lincoln Univ.). Vice-Prin., State Normal School, Elizabeth City, N. C.

BLUE, A. H., M.S. (Iowa). Grad. Asst., Univ. of Iowa, Iowa City, Ia.

BRIXEY, J. C., A.M. (Oklahoma). Instr., Univ. of Oklahoma, Norman, Okla.

BULLITT, W. M., B.S. (Princeton), LL.B. (Louisville). 1710–1726 Inter-Southern Bldg., Louisville, Ky.

CAMPBELL, W. B., B.S. in M.E. (Penna.). Instr., Cornell Univ., Ithaca, N. Y.

COOKE, J. C., A.M. (Columbia). Prof., Math. and Physics, St. Ambrose College, Davenport, Ia.

CUMMING, FORREST, A.M. (Georgia). Adj. Prof., Univ. of Georgia, Athens, Ga.

DUERKSEN, J. A., A.B. (Pomona). Junior Mathematician, Coast and Geod. Survey, Washington, D. C.

DUREN, W. L., Jr. Undergrad. Asst., Tulane Univ., New Orleans, La.

EDMONDSON, AUBREY, B.S. in C.E. (Miss. A. and M. Coll.). Instr., A. & M. Coll., Starksville Miss.

ERICKSON, E. E., M.S. (Iowa). Asst. Prof., Miami Univ., Oxford, Ohio.

EVERETT, OTIS, A.B. (Harvard), LL.B. (Fordham) 57 Broadway, New York, N. Y.

FENTRESS, G. L., M.S. (Va. Polytech). Jr. Mathematician, Coast and Geod. Survey, Washington, D. C.

FORNO, DORA M., A.M. (Tulane). N. O. Normal School, New Orleans, La.

FRANKENBUSH, BERTHA E., A.M. (Tulane). Teacher, Miller's High School, New Orleans, La.

GIBSON, E. W., Jr., A.B. (Norwich). Asst. Mathematician, Coast and Geod. Survey, Washington, D. C.

HAMPTON, LAURENCE, B.S. (Beloit). Instr., Colorado School of Mines, Golden, Colorado.

HANSEN, G. W., (Whitman). Mathematician, Coast and Geod. Survey, Washington, D. C.

HAWKES, JULIA M., Ph.D. (Michigan). Prof., Math. and Astr., Doane College, Crete, Nebr.

JIMENEZ, REV. FELIPE. Senior Student, Notre Dame Univ., Notre Dame, Ind.

JOHNSON, EDITH R., B.S. (Chicago). 2428 Tola Ave., Pasadena, Calif.

KAUFMAN, H. P., M.E. (Polytech. Inst. of Brooklyn). Jr. Mathematician, Coast and Geodetic Survey, Washington, D. C.

KILLEN, C. G., A.B. (La. State Normal College). Asst. Prof., La. State Normal College, Natchitoches, La.

KNIGHT, KATE K., A.B. (East Central Teachers Coll.). Instr., East Central Teachers College, Ada, Okla.

LOCKWOOD, E. C., Ph.B. (Brown). Grad. Student, Brown Univ., Providence, R. I.

MACDONALD, MARY H., A.M. (Bryn Mawr). Teacher, Physics and Geom., Shipley School, Bryn Mawr, Pa.

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